

Slow Modes in Passive Advection

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The anomalous scaling in the Kraichnan model of advection of the passive scalar by a random velocity field with nonsmooth spatial behavior is traced to the presence of slow resonance-type collective modes of the stochastic evolution of fluid trajectories. We show that the slow modes are organized into infinite multiplets of descendants of the primary conserved modes. Their presence is linked to the nondeterministic behavior of the Lagrangian trajectories at high Reynolds numbers caused by the sensitive dependence on initial conditions within the viscous range where the velocity fields are more regular. Revisiting the Kraichnan model with smooth velocities, we describe the explicit solution for the stationary state of the scalar. The properties of the probability distribution function of the smeared scalar in this state are related to a quantum mechanical problem involving the Calogero–Sutherland Hamiltonian with a potential.

KEY WORDS: Passive scalar; Lagrangian trajectories; anomalous scaling.

1. INTRODUCTION

One of the basic open problems in fully developed hydrodynamical turbulence is the understanding of the origin of observed violations of the Kolmogorov⁽¹⁷⁾ scaling. The violations indicate presence of strong short-distance intermittency in the turbulent cascade, i.e. of frequent occurrence of large fluctuations on short distances. Recently some progress has been achieved in the understanding of the analogous problem for the passive advection of a scalar quantity by a random velocity field. The scalar is known to exhibit strong short-distance intermittency even if such is absent

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in the velocity field. In the simplest model of the passive scalar, due to Kraichnan,⁽¹⁸⁾ one assumes a Gaussian distribution of time-decorrelated and spatially nonsmooth velocities. The anomalous scaling of the scalar in this model was related in refs. 7, 15, and 24 to zero modes of differential operators describing the stochastic evolution of the flow. In the present paper we elaborate on this idea showing that the short-distance intermittency of the scalar is due to the presence of slow collective modes in the otherwise super-diffusive evolution of the (quasi)-Lagrangian trajectories of fluid particles. We show that in the Kraichnan model the slow modes, reminiscent of resonances in multi-body problems, are organized into infinite multiplets of descendants with the zero modes playing the role of primary objects. This structure might indicate the presence of hidden infinite symmetries in the Kraichnan problem.

The other important feature of the Lagrangian flow in nonsmooth velocities is its intrinsically probabilistic character: the Lagrangian trajectories of the fluid particles behave randomly even in a fixed velocity field. This phenomenon appears to be closely related to the presence of the slow modes in the stochastic flow of fluid particles. In more realistic velocity fields which are regularized on the viscous scale the effective stochasticity of the fluid trajectories is due to their sensitive dependence on initial conditions on scales shorter than the viscous one. We expect both phenomena: the presence of resonant slow modes in the Lagrangian flow and the non-deterministic character of the fluid trajectories, to be present in more realistic high Reynolds number velocity ensembles and to be responsible for their intermittence.

The version of the Kraichnan model with smooth velocity fields, relevant for the description of the distances smaller than the viscous scale, has been intensively studied, see refs. 19, 24 and 25 to 10. We return to this case developing further the tools of harmonic analysis used first for this model in refs. 24 and 26. These tools allow a fast calculation of the the Lyapunov exponents for the flow of fluid particles found first in refs. 6 and 8. We also explicitly construct the stationary state of the scalar relating its functional Fourier transform to a certain Schrödinger operator on the symmetric space $SL(d)/SO(d)$ where d is the space dimension. In particular, we compute the exponential decay rate of the probability distribution functions $p(\theta)$ of smeared scalar values obtaining in three dimensions (and above) a result different from that of refs. 6, 8, and 10. The discrepancy is traced to the contribution of correlations of different (pairs of) fluid trajectories disregarded in refs. 6, 8, and 10. For the exponential decay rate of the Fourier transform of $p(\theta)$ our results reproduce fully the calculations of refs. 6 and 8 and confirm their semiclassical interpretation proposed in ref. 10.

The paper is organized as follows. In Sect. 2 we present the Kraichnan model and obtain its solution employing a path integral formalism. Section 3 recalls briefly the analysis of refs. 3 and 15 establishing anomalous scaling of the scalar by perturbative analysis of the scaling zero modes of operators governing the flow. The physical interpretation of the zero modes as scaling structures conserved in mean is the subject of Sect. 4. Section 5 discusses the collective slow modes of the random flow of fluid particles. The analytic origin of the slow modes is unraveled in more technical Sect. 6. Section 7 describes the intricacies of the probabilistic description of fluid trajectories. Finally, Sects. 8 and 9 study in detail the case of Kraichnan model with smooth velocity field elaborating on the earlier results of ref. 24 and of ref. 25 to ref. 10. Appendix A explicitly analyzes the slow modes in the relative motion of two fluid particles in nonsmooth velocity field. Appendix B contains some more details on the smooth velocity case related to the results of refs. 1 and 6.

2. KRAICHNAN MODEL OF PASSIVE SCALAR

Let us consider an advection of a scalar quantity $T(t, x)$ (the temperature) in d space dimensions. The time evolution of T is governed by the linear equation

$$\partial_t T + v \cdot \nabla T - \kappa \Delta T = f \quad (2.1)$$

where $v(t, x)$ is the incompressible ($\nabla \cdot v = 0$) velocity field of the advecting fluid, κ is the diffusion constant and $f(t, x)$ is a given source term. Denote by $R(t, x; t_0, x_0)$ the solution of the homogeneous equation

$$(\partial_t + v \cdot \nabla - \kappa \Delta) R(t, t_0) = 0 \quad (2.2)$$

with the initial condition

$$R(t_0, x; t_0, x_0) = \delta(x - x_0) \quad (2.3)$$

We shall call $R(t, x; t_0, x_0)$ the evolution kernel and the corresponding operator $R(t, t_0)$ the evolution operator. The solution of Eq. (2.1) has the form

$$T(t, x) = \int R(t, x; t_0, y) T(t_0, y) dy + \int_{t_0}^t \int R(t, x; s, y) f(s, y) ds dy \quad (2.4)$$

with $T(t_0)$ being the initial configuration of T at time t_0 .

There exists a functional integral formula for the evolution kernel which, for sufficiently regular v , may be easily given a rigorous sense as an integral with respect to the Wiener measure with density:

$$R(t, x; t_0, x_0) = \int_{\substack{x(t_0) = x_0 \\ x(t) = x}} e^{-(1/4\kappa) \int_{t_0}^t ds [\dot{x}(s) - v(s, x(s))]^2} D\mathbf{x} \quad (2.5)$$

where $\dot{x} \equiv (dx/dt)$. It will be useful to rewrite the above functional integral as a phase space one:

$$R(t, x; t_0, x_0) = \int_{\substack{x(t_0) = x_0 \\ x(t) = x}} e^{-\int_{t_0}^t ds [\kappa p(s)^2 + ip(s) \cdot (\dot{x}(s) - v(s, x(s)))]} D\mathbf{x} D\mathbf{p} \quad (2.6)$$

with the Gaussian integral over the unconstrained paths $s \mapsto p(s)$ reproducing the previous integral. From the functional integral representations it is clear that when $\kappa \rightarrow 0$ then

$$R(t, x; t_0, x_0) \rightarrow \delta(x - x(t; t_0, x_0)) \quad (2.7)$$

where $x(t; t_0, x_0)$ is the Lagrangian trajectory of the fluid particle satisfying the equations

$$\dot{x} = v(t, x), \quad x(t_0) = x_0 \quad (2.8)$$

Indeed, we may set $\kappa = 0$ in the phase space integral (2.6) and the p -integral gives then a delta function(al) concentrated on the Lagrangian trajectory. For small positive κ , on the other hand, $R(t, x; t_0, x_0)$ is the probability distribution function (p.d.f.) of the endpoint of a small Brownian motion around the Lagrangian trajectory. Such a Brownian motion $x_\beta(t; t_0, x_0)$ with a drift is a solution of the stochastic ordinary differential equation (ODE)

$$dx = v(t, x) dt + \kappa d\beta, \quad x(t_0) = x_0 \quad (2.9)$$

with $\beta(t)$ denoting the Brownian motion without drift. Thus

$$R(t, x; t_0, x_0) = E(\delta(x - x_\beta(t; t_0, x_0))) \quad (2.10)$$

where $E(\cdot)$ denotes the expectation with respect to the Wiener measure of β . Equation (2.10) is another form of Eq. (2.5).

We shall be interested in the situation when both velocities v and source f in Eq. (2.1) are random so that Eq. (2.1) is a stochastic PDE.

In order to solve such a stochastic equation, we should define the evolution kernel $R(t, x; t_0, x_0)$ as a random, v -dependent process or, in plain English, be able to compute various expectation values of R 's like

$$\mathcal{P}_n(\mathbf{t}, \mathbf{x}; \mathbf{t}_0, \mathbf{x}_0) \equiv \left\langle \prod_{i=0}^n R(t_i, x_i; t_{0,i}, x_{0,i}) \right\rangle = E \left(\prod_{i=0}^n \delta(x_i - x_{\beta_i}(t_i; t_{0,i}, x_{0,i})) \right) \tag{2.11}$$

where $\mathbf{t} \equiv (t_1, \dots, t_n)$, $\mathbf{x} \equiv (x_1, \dots, x_n)$ and similarly for $\mathbf{t}_0, \mathbf{x}_0$ and where $E(\cdot)$ denotes the expectation w.r.t. β_i 's and v . It is clear from the second expression for $\mathcal{P}_n(\mathbf{t}, \mathbf{x}; \mathbf{t}_0, \mathbf{x}_0)$ that it gives the joint p.d.f. of the ends $x_{\beta_i}(t_i; t_{0,i}, x_{0,i})$ of n Brownian motions (independent for given v) around the Lagrangian trajectories starting at times $t_{0,i}$ from points $x_{0,i}$. The $\kappa \rightarrow 0$ limit of $\mathcal{P}_n(\mathbf{t}, \mathbf{x}; \mathbf{t}_0, \mathbf{x}_0)$ (if exists) should simply give the joint p.d.f. of the endpoints $x(t_i; t_{0,i}, x_{0,i})$ of n Lagrangian trajectories.⁵

Let us assume that the velocity is a Gaussian stationary field with mean zero and covariance

$$\langle v^\alpha(t_1, x_1) v^\beta(t_2, x_2) \rangle = D^{\alpha\beta}(t_{12}, x_{12}) \tag{2.12}$$

where $t_{12} \equiv t_1 - t_2$, $x_{12} \equiv x_1 - x_2$ and $\partial_\alpha D^{\alpha\beta}(t, x) = 0$ to assure the incompressibility. Employing the phase space path integral representation (2.6) and performing the Gaussian functional integration over v , we obtain⁶

$$\begin{aligned} \mathcal{P}_n(\mathbf{t}, \mathbf{x}; \mathbf{t}_0, \mathbf{x}_0) &= \left\langle \int_{\substack{x_i(t_{0,i}) = x_{0,i} \\ x_i(t_i) = x_i}} e^{-\sum_i \{ \int_{t_{0,i}}^{t_i} ds_i [\kappa p_i(s_i)^2 + ip_i(s_i) \cdot (\dot{x}_i(s_i) - v(s_i, x_i(s_i)))] \}} D\mathbf{x} D\mathbf{p} \right\rangle \\ &= \int_{\substack{x_i(t_{0,i}) = x_{0,i} \\ x_i(t_i) = x_i}} e^{-\sum_i \{ \int_{t_{0,i}}^{t_i} ds_i [\kappa p_i(s_i)^2 + ip_i(s_i) \cdot \dot{x}_i(s_i)] \}} \\ &\quad \times e^{-1/2 \sum_{i,j} \int_{t_{0,i}}^{t_i} ds_i \int_{t_{0,j}}^{t_j} ds_j D^{\alpha\beta}(s_i - s_j, x_i(s_i) - x_j(s_j)) p_i^\alpha(s_i) p_j^\beta(s_j)} D\mathbf{x} D\mathbf{p} \end{aligned} \tag{2.13}$$

If, following Kraichnan,⁽¹⁸⁾ we assume that $v(t, x)$ is also decorrelated in time, i.e., that

$$D^{\alpha\beta}(t, x) = \delta(t) \mathcal{D}^{\alpha\beta}(x) \tag{2.14}$$

⁵ K.G. thanks Ya. Sinai for attracting his attention to the statistics of Lagrangian trajectories.

⁶ Similar expressions appeared in refs. 4 and 9.

then formula (2.13) for $\mathcal{P}_n(\mathbf{t}, \mathbf{x}; \mathbf{t}_0, \mathbf{x}_0)$ may be further simplified. Let us set all t_i equal to t and all t_{0i} equal to t_0 . We shall denote the corresponding \mathcal{P}_n by $\mathcal{P}_n(t, \mathbf{x}; t_0, \mathbf{x}_0)$ (the general case can be reconstructed from the special one for v 's delta-correlated in time). The fundamental property of the p.d.f.'s $\mathcal{P}_n(t, \mathbf{x}; t_0, \mathbf{x}_0)$ for the time-decorrelated velocities (not necessarily Gaussian) is the composition property

$$\int \mathcal{P}_n(t, \mathbf{x}; s, \mathbf{y}) \mathcal{P}_n(s, \mathbf{y}; t_0, \mathbf{x}_0) d\mathbf{y} = \mathcal{P}_n(t, \mathbf{x}; t_0, \mathbf{x}_0) \quad (2.15)$$

From expression (2.13) we obtain, assuming relation (2.14),

$$\begin{aligned} & \mathcal{P}_n(t, \mathbf{x}; t_0, \mathbf{x}_0) \\ &= \int_{\substack{x_i(t_0) = x_{0,i} \\ x_i(t) = x_i}} e^{-\int_{t_0}^t ds [\kappa \sum_i p_i(s)^2 + (1/2) \sum_{i,j} \mathcal{D}^{\alpha\beta}(x_i(s) - x_j(s)) p_i^\alpha(s) p_j^\beta(s) + i \sum_i p_i(s) \cdot \dot{x}_i(s)]} \\ & \quad \times D\mathbf{x} D\mathbf{p}. \end{aligned} \quad (2.16)$$

It is easy to see that the right hand side is a phase space path integral expression for the heat kernel (dynamical Green function) of the 2nd order (positive, elliptic) differential operator

$$\mathcal{M}_n = -\frac{1}{2} \sum_{i,j=1}^n \mathcal{D}^{\alpha\beta}(x_{ij}) \partial_{x_i^\alpha} \partial_{x_j^\beta} - \kappa \sum_{i=1}^n \Delta_{x_i} \quad (2.17)$$

i.e., that

$$\mathcal{P}_n(t, \mathbf{x}; t_0, \mathbf{x}_0) = e^{-(t-t_0) \mathcal{M}_n(\mathbf{x}, \mathbf{x}_0)} \quad (2.18)$$

compare Eq. (2.6). Note that, due to incompressibility, there is no ordering ambiguity in passing from the path integral to the expression for \mathcal{M}_n . Rigorously minded person may take expressions (2.18) as defining the evolution operators for the stochastic PDE equation (2.1) in the time decorrelated case. Of course, the composition property (2.15) follows from the semigroup law for the heat kernels.

Let us now go back to the passive scalar. Assume that both v and f are independent stationary processes. Imposing also the zero initial condition for T at $t_0 = -\infty$, we obtain using Eq. (2.4) the following expression for the correlators of T :

$$\left\langle \prod_{i=0}^n T(t_i, x_i) \right\rangle = \prod_{i=1}^n \int_{-\infty}^{t_i} ds_i \int dy_i \mathcal{P}_n(\mathbf{t}, \mathbf{x}; \mathbf{s}, \mathbf{y}) \left\langle \prod_{i=1}^n f(s_i, y_i) \right\rangle \quad (2.19)$$

It should be clear that if a stationary state of the scalar is generated for large time and independent of the initial conditions (say, decaying at infinity) then its correlation functions should be given by the above equation. Hence the importance of understanding the behavior of the p.d.f.'s $\mathcal{P}_n(\mathbf{t}, \mathbf{x}; \mathbf{t}_0, \mathbf{x}_0)$.

Assume now that the source f (independent of v) is a Gaussian process with mean zero and covariance

$$\langle f(t_1, x_1) f(t_2, x_2) \rangle = \delta(t_{12}) \mathcal{C}(x_{12}) \tag{2.20}$$

where \mathcal{C} is a positive definite test function. In this case and for the Gaussian, time decorrelated velocities, Eqs. (2.19) simplify permitting an inductive calculation of the stationary correlation functions of the scalar. Let us see how this works for equal time correlators. We may consider only the even-point functions of T , $\mathcal{F}_{2n}(\mathbf{x}) \equiv \langle T(t, x_1) \cdots T(t, x_{2n}) \rangle$, since the odd correlators of f vanish implying the same property of the T correlators. For the 2-point function we obtain

$$\begin{aligned} \mathcal{F}_2(x_{12}) &= \int_{-\infty}^t ds \int d\mathbf{y} e^{-(t-s)} \mathcal{M}_2(\mathbf{x}, \mathbf{y}) \mathcal{C}(y_{12}) \\ &= \int d\mathbf{y} \cdot \mathcal{M}_2^{-1}(\mathbf{x}, \mathbf{y}) \mathcal{C}(y_{12}) \end{aligned} \tag{2.21}$$

and for the 4-point function

$$\begin{aligned} \mathcal{F}_4(\mathbf{x}) &= \sum_{1 \leq i < j \leq 4} \int_{-\infty}^t ds \int_{-\infty}^s ds' \int d\mathbf{y} \int d\mathbf{z} e^{-(t-s)} \mathcal{M}_4(\mathbf{x}, \mathbf{y}) \mathcal{C}(y_{ij}) \\ &\quad \times e^{-(s-s')} \mathcal{M}_2(y_{1, \overline{ij}}, y_4, \mathbf{z}) \mathcal{C}(z_{12}) \\ &= \sum_{1 \leq i < j \leq 4} \int \mathcal{M}_4^{-1}(\mathbf{x}, \mathbf{y}) \mathcal{F}_2(y_{1, \overline{ij}}, y_4) \mathcal{C}(y_{ij}) d\mathbf{y} \end{aligned} \tag{2.22}$$

Similar arguments for the $2n$ -point function give

$$\mathcal{F}_{2n}(\mathbf{x}) = \sum_{1 \leq i < j \leq 2n} \int \mathcal{M}_{2n}^{-1}(\mathbf{x}, \mathbf{y}) \mathcal{F}_{2n-2}(y_{1, \overline{ij}}, y_{2n}) \mathcal{C}(y_{ij}) d\mathbf{y} \tag{2.23}$$

The above equations permit an inductive calculation of the stationary equal time correlation functions of T with the use of the (static) Green functions $\mathcal{M}_n^{-1}(\mathbf{x}, \mathbf{y})$ of operators \mathcal{M}_n .

3. ZERO-MODE DOMINANCE

We shall be interested in the case where the spatial part $\mathcal{D}^{\alpha\beta}(x)$ of the v -covariance has the form

$$\mathcal{D}^{\alpha\beta}(x) = \mathcal{D}_0 \delta^{\alpha\beta} - d^{\alpha\beta}(x) \quad (3.1)$$

where $d^{\alpha\beta}(x)$ scales with power $2 - \gamma$,

$$d^{\alpha\beta}(x) \sim |x|^{2-\gamma} \quad (3.2)$$

for small $|x|$. Here $0 \leq \gamma \leq 2$ is a fixed parameter. The tensorial form of $d^{\alpha\beta}(x)$ is fixed for small $|x|$ by the incompressibility condition $\partial_\alpha d^{\alpha\beta}(x) = 0$:

$$d^{\alpha\beta}(x) \cong \frac{D}{d-1} ((d+1-\gamma) \delta^{\alpha\beta} |x|^{2-\gamma} - (2-\gamma) x^\alpha x^\beta |x|^{-\gamma}) \equiv d_{sc}^{\alpha\beta}(x) \quad (3.3)$$

where D is a constant. For $0 < \gamma < 2$, one may take

$$\mathcal{D}^{\alpha\beta}(x) \sim \int \frac{e^{-ik \cdot x}}{(k^2 + m^2)^{(d+2-\gamma)/2}} \left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right) dk \quad (3.4)$$

where m is an infrared regulator. Relations (3.2) and (3.3) hold then for $m|x| \ll 1$. When $m \rightarrow 0$, $d^{\alpha\beta}(x)$ tends to the scaling form $d_{sc}^{\alpha\beta}(x)$ but \mathcal{D}_0 diverges like $\mathcal{O}(m^{\gamma-2})$.

The Gaussian distribution with covariance given by Eqs. (2.14) and (3.1) is relatively far from a realistic description of the statistics of turbulent flows. First, it excludes the velocity intermittency, i.e., more frequent occurrence of large deviations of velocity differences than in the normal distribution. Such occurrence characterizes short scales in the inertial interval of the turbulent cascade. Second, the time decorrelation is a brutal approximation since one observes scale-dependent time correlations in turbulent flows. The power-law growth of the velocity difference covariance

$$\begin{aligned} \langle (v^\alpha(t_1, x_1) - v^\alpha(t_1, x_2))(v^\beta(t_2, x_1) - v^\beta(t_2, x_2)) \rangle &= 2\delta(t_{12}) d^{\alpha\beta}(x_{12}) \\ &\sim \delta(t_{12}) |x_{12}|^{2-\gamma} \end{aligned} \quad (3.5)$$

mimics, however, the expected behavior in the turbulent cascade (the Kolmogorov value of the scaling exponent corresponds to $\gamma = \frac{2}{3}$ since time appears to scale like length to power γ in the model). The point is that even the velocity distributions far from realistic, as the one described above, induce strongly intermittent scalar distributions and the purpose of the

Kraichnan model is to understand this phenomenon in the simplest context.

Operators \mathcal{M}_n may be rewritten in the form

$$\mathcal{M}_n = \sum_{1 \leq i < j \leq n} d^{\alpha\beta}(x_{ij}) \partial_{x_i^\alpha} \partial_{x_j^\beta} - \kappa \sum_{i=1}^n \Delta_{x_i} - \frac{1}{2} \mathcal{D}_0 \left(\sum_{i=1}^n \partial_{x_i^2} \right)^2 \quad (3.6)$$

where the last term drops out in the action on translationally invariant functions. We shall, somewhat pedantically, denote the operator \mathcal{M}_n acting in the translational invariant sector by M_n . We shall view M_n as an operator in the reduced space $L^2(\mathbf{R}^{d_n})$, with $d_n \equiv (n-1)d$. This is the space of functions of the difference variables $x_{in} \equiv x_i - x_n$, square-integrable with the measure $d'\mathbf{x} \equiv dx_{1n} \cdots dx_{(n-1)n}$. The heat kernel of M_n ,

$$e^{-(t-t_0)M_n}(\mathbf{x}, \mathbf{x}_0) = \int_{\mathbf{R}^d} \mathcal{P}_n(t, \mathbf{x} + \mathbf{a}; t_0, \mathbf{x}_0) da \equiv P_n(t, \mathbf{x}; t_0, \mathbf{x}_0) \quad (3.7)$$

with $\mathbf{a} \equiv (a, \dots, a)$, gives the joint p.d.f. of the differences x_{in} of the Lagrangian trajectories starting at points \mathbf{x}_0 (or, equivalently, the joint p.d.f. of the Lagrangian trajectories in the quasi-Lagrangian picture⁽²⁾). It is translationally invariant separately in \mathbf{x} and \mathbf{x}_0 .

In the limit $m \rightarrow 0$ when $d^{\alpha\beta}(x)$ takes the scaling form (3.3) but \mathcal{D}_0 diverges, operator M_n , unlike \mathcal{M}_n tends to the limit which for $\kappa = 0$ coincides with the scaling operator

$$M_n^{\text{sc}} = \sum_{i \leq j < k \leq n} d_{\text{sc}}^{\alpha\beta}(x_{ij}) \partial_{x_i^\alpha} \partial_{x_j^\beta} \quad (3.8)$$

of scaling dimension $-\gamma$. M_n^{sc} is a positive singular elliptic differential operator of the 2nd order in $L^2(\mathbf{R}^{d_n})$. By a simple self-consistent analysis one may convince oneself that, at least for γ close to 2, $e^{-tM_n}(\mathbf{x}, \mathbf{y})$ and $M_n^{-1}(\mathbf{x}, \mathbf{y})$ converge pointwise when $m \rightarrow 0$ and $\kappa \rightarrow 0$ to the heat kernel $e^{-tM_n^{\text{sc}}}(\mathbf{x}, \mathbf{y})$ and the Green function $(M_n^{\text{sc}})^{-1}(\mathbf{x}, \mathbf{y})$, respectively. The latter should satisfy bounds that may be inferred from a semi-classical analysis of the path integral expressions (2.16) with $\mathcal{D}^{\alpha\beta}$ replaced by $d_{\text{sc}}^{\alpha\beta}$ and κ set to zero.⁷ In the limit $\gamma \rightarrow 2$, $d_{\text{sc}}^{\alpha\beta}(x)$ tends for non-zero x to a constant times $\delta^{\alpha\beta}$ and M_n^{sc} becomes proportional to the d_n -dimensional Laplacian. When γ is close to 2, the heat kernel $e^{-tM_n^{\text{sc}}}(\mathbf{x}, \mathbf{y})$ and the Green function $(M_n^{\text{sc}})^{-1}(\mathbf{x}, \mathbf{y})$ differ little from the heat kernel and the Green function of

⁷ To our knowledge, such bounds have not been obtained in the mathematical literature and constitute an open mathematical problem.

the Laplacian. In particular, $e^{-tM_n^{\text{sc}}}(\mathbf{x}, \mathbf{y})$ is finite everywhere and $(M_n^{\text{sc}})^{-1}(\mathbf{x}, \mathbf{y})$ is infinite only when $\mathbf{x} = \mathbf{y}$. We expect this to hold for all $\gamma > 0$. When $2 - \gamma$ is small, the behaviors of $(M_n^{\text{sc}})^{-1}(\mathbf{x}, \mathbf{y})$ around $\mathbf{x} = \mathbf{y}$ and at infinity differ from those of the Green function of the d_n -dimensional Laplacian by $\mathcal{O}(2 - \gamma)$ modifications of the power laws. All that implies that the pointwise limits $m \rightarrow 0$ and $\kappa \rightarrow 0$ of the equal time correlators of T given by Eqs. (2.23) exist, at least for γ close to 2, and are given by the version of the same equations employing the scaling Green functions $(M_n^{\text{sc}})^{-1}(\mathbf{x}, \mathbf{y})$ (with $d\mathbf{y}$ replaced by $d'\mathbf{y}$). From now on, we shall deal only with these limits and with the scaling operators M_n^{sc} and shall drop the superscript “sc.”

We are interested in the behavior of the equal time correlators of T , especially in their scaling properties, in the situation when the source acts only on large distances, i.e., in our Gaussian model, when the spatial part \mathcal{C} of the covariance of f is almost constant. We may study this regime by replacing $\mathcal{C}(x)$ by $\mathcal{C}_L(x) \equiv \mathcal{C}(x/L)$ and by examining the large L behavior of the equal time correlators $\mathcal{F}_{2n} \equiv \mathcal{F}_{2n, L}$. The following result, which may be referred to as the *zero mode dominance*, has been described in refs. 3 and 15: at γ sufficiently close to 2 and at $m, \kappa = 0$,

$$\mathcal{F}_{2n, L}(\mathbf{x}) = A_\varphi L^{\rho_{2n}} \mathcal{F}_{2n}^0(\mathbf{x}) + \mathcal{O}(L^{-2 + \mathcal{O}(2 - \gamma)}) + [\dots] \quad (3.9)$$

for $n > 1$. Above, A_φ is a non-universal amplitude (a constant depending on the shape of the source covariance \mathcal{C}) and $\rho_{2n} = (2n(n - 1)/(d + 2))(2 - \gamma) + \mathcal{O}((2 - \gamma)^2)$ is a universal (i.e., \mathcal{C} -independent) anomalous exponent. \mathcal{F}_{2n}^0 is the scaling (translationally invariant) zero mode of M_{2n} ,

$$\mathcal{F}_{2n}^0(\lambda \mathbf{x}) = \lambda^{\gamma n - \rho_{2n}} \mathcal{F}_{2n}^0(\mathbf{x}), \quad M_{2n} \mathcal{F}_{2n}^0 = 0. \quad (3.10)$$

$[\dots]$ denotes terms which do not depend on at least one x_i and as such do not contribute to the correlation functions of scalar differences $\langle \prod_i (T(t, x_i) - T(t, y_i)) \rangle$.

$$\mathcal{F}_{2n}^0(\mathbf{x}) = \mathcal{S} x_{12}^2 x_{34}^2 \dots x_{2n-1, 2n}^2 + \mathcal{O}(2 - \gamma) + [\dots] \quad (3.11)$$

where \mathcal{S} is the symmetrization operator. The contribution to \mathcal{F}_{2n}^0 proportional to $2 - \gamma$ is also known up to $[\dots]$ terms.⁽³⁾ A similar analysis was performed in refs. 5 and 7 for large space dimensions d .

The main implication of the relation (3.9) is the anomalous scaling of the $n > 1$, γ close to 2 structure functions $S_{2n, L}(x) \equiv \langle (T(t, x) - T(t, 0))^{2n} \rangle$. At $m, \kappa = 0$ and for $|x|/L \ll 1$,

$$S_{2n, L}(x) \sim L^{\rho_{2n}} |x|^{\gamma n - \rho_{2n}} \quad (3.12)$$

The above behavior contradicts the simple dimensional prediction $S_{2n}(x) \sim |x|^m$ which holds only for the 2-point function.

Let us sketch the argument leading to the result (3.9), based on applying the Mellin transform to select the dominant contributions for large L . It will be convenient to work with a version of operators M_n of scaling dimension zero

$$N_n = R_n^{\gamma/2} M_n R_n^{\gamma/2} \tag{3.13}$$

where $R_n^2 \equiv \sum_{i < j} (x_i - x_j)^2$. N_n is also a positive (unbounded) operator⁸ in $L^2(\mathbf{R}^{d_n})$. Since it commutes with the self-adjoint generator of dilations

$$D_n = \frac{1}{i} \left(\sum_i x_i^\alpha \partial_{x_i^\alpha} + \frac{d_n}{2} \right) \tag{3.14}$$

it is partially diagonalized by the Mellin transform of the translationally invariant functions

$$f(\mathbf{x}) \rightarrow \hat{f}(\sigma, \hat{\mathbf{x}}) = \int_0^\infty \lambda^{-\sigma-1} f(\lambda \hat{\mathbf{x}}) d\lambda \tag{3.15}$$

The map (3.15) is a unitary transformation, diagonalizing D_n , between

$$L^2(\mathbf{R}^{d_n}) \quad \text{and} \quad L^2 \left(\left\{ \text{Re } \sigma = -\frac{d_n}{2} \right\} \right) \otimes L^2(S^{d_n-1})$$

where S^{d_n-1} is composed of points $\hat{\mathbf{x}} = \mathbf{x}/R_n$ in the space \mathbf{R}^{d_n} of difference variables. In the language of the Mellin transform, N_n becomes a family $\hat{N}_n(\sigma)$ of operators in $L^2(S^{d_n-1})$. In particular,

$$(N_n^{-1} f)^\wedge(\sigma, \hat{\mathbf{x}}) = \int_{S^{d_n-1}} \hat{N}_n^{-1}(\sigma; \hat{\mathbf{x}}, \hat{\mathbf{y}}) \hat{f}(\sigma, \hat{\mathbf{y}}) d\hat{\mathbf{y}} \tag{3.16}$$

where the Mellin-transformed Green function $\hat{N}_n^{-1}(\sigma; \hat{\mathbf{x}}, \hat{\mathbf{y}})$ satisfies the hermiticity relation

$$\overline{\hat{N}_n^{-1}(\sigma; \hat{\mathbf{y}}, \hat{\mathbf{x}})} = \hat{N}_n^{-1}(-d_n - \bar{\sigma}; \hat{\mathbf{x}}, \hat{\mathbf{y}}) \tag{3.17}$$

⁸ Technically, N_n , as well as M_n , may be defined as the Friedrichs extension of its restriction to smooth functions with compact support, vanishing around the diagonals $x_i = x_j$.

It is a meromorphic function of σ with simple poles for generic γ . Around the poles

$$\hat{N}_n^{-1}\left(\sigma - \frac{\gamma}{2}; \hat{\mathbf{x}}, \hat{\mathbf{y}}\right) \cong \frac{1}{\sigma - \sigma_i} f_i(\hat{\mathbf{x}}) \overline{g_i(\hat{\mathbf{y}})} \quad (3.18)$$

where f_i are the scaling zero modes of M_n of scaling dimension σ_i and g_i are similar modes with scaling dimensions $-d_n + \gamma - \sigma_i$, both in $L^2(S^{d_n-1})$. Although operator M_n has continuous spectrum when considered as a positive operator in $L^2(\mathbf{R}^{d_n})$, it induces an operator $\hat{N}_n(\sigma - (\gamma/2))$ in $L^2(S^{d_n-1})$ with a discrete spectrum when acting on scaling functions with a scaling dimension σ . The scaling zero modes occur at discrete values σ_i of σ for which zero belongs to the spectrum.

It is easy to see from the inductive equations (2.23) that

$$\mathcal{F}_{2n, L}(\mathbf{x}) = L^{n\gamma} \mathcal{F}_{2n, 1}(\mathbf{x}/L) \quad (3.19)$$

and that, with the use of the Mellin transform, these equations may be rewritten as

$$\begin{aligned} \mathcal{F}_{2n, L}(\mathbf{x}) &= L^{n\gamma} \int_{\text{Re } \sigma = -(d_n/2) + (\gamma/2)} \frac{d\sigma}{2\pi i} (R_n/L)^\sigma \hat{N}_n^{-1}\left(\sigma - \frac{\gamma}{2}; \hat{\mathbf{x}}, \hat{\mathbf{y}}\right) \\ &\quad \times (\mathcal{F}_{2n-2, 1} \otimes \mathcal{C})^\wedge(\sigma - \gamma, \hat{\mathbf{y}}) d\hat{\mathbf{y}} \end{aligned} \quad (3.20)$$

Shifting the integration contour to $\text{Re } \sigma = \gamma n + 2 - \mathcal{O}(2 - \gamma)$, we obtain

$$\begin{aligned} \mathcal{F}_{2n, L}(\mathbf{x}) &= - \sum_i L^{\gamma n - \sigma_i} R_n^{\sigma_i} \int \text{Res}_{\sigma = \sigma_i} \hat{N}_n^{-1}\left(\sigma - \frac{\gamma}{2}; \hat{\mathbf{x}}, \hat{\mathbf{y}}\right) \\ &\quad \times (\mathcal{F}_{2n-2, 1} \otimes \mathcal{C})^\wedge(\sigma - \gamma, \hat{\mathbf{y}}) d\hat{\mathbf{y}} + \mathcal{O}(L^{-2 + \mathcal{O}(2 - \gamma)}) \end{aligned} \quad (3.21)$$

where the sum runs over the poles σ_i in the strip

$$-\frac{d_n}{2} + \frac{\gamma}{2} < \text{Re } \sigma_i < \gamma n + 2 - \mathcal{O}(2 - \gamma) \quad (3.22)$$

and the last term, suppressed for large L , comes from the shifted contour. There are two types of poles: those coming from $(\mathcal{F}_{2n-2, 1} \otimes \mathcal{C})^\wedge$ and those in the Green function \hat{N}_n^{-1} . The first ones contribute either to $[\dots]$ or to $\mathcal{O}(L^{-2 + \mathcal{O}(2 - \gamma)})$ in Eq. (3.9) and are not interesting for us (at least for γ close to 2). The second ones are related to the scaling zero modes of M_n ,

see Eq. (3.18). Only rotationally invariant (if \mathcal{C} has the same property) zero modes symmetric under permutations of points and square-integrable on S^{d_n-1} contribute to $\mathcal{F}_{2n,L}$. Such zero modes may be studied for γ close to 2 by perturbative analysis of discrete-spectrum operators M_n acting on scaling functions or, equivalently, of operators $\hat{N}_n(\sigma)$ acting in $L^2(S^{d_n-1})$. (Recall that for $\gamma=2$, M_n becomes the d_n -dimensional Laplacian). The result is that, for γ close to 2, all but one zero modes in the strip (3.22) contribute [\dots] terms. \mathcal{F}_{2n}^0 is the exception and it has scaling dimension $\sigma_0 = \gamma n - \rho_{2n}$. We expect essentially the same picture with the zero mode domination of correlation functions to persist for all $\gamma > 0$. One of the open problems is whether there are other non-[\dots] zero modes entering the strip (3.22) for smaller γ and whether, if they cross, they may produce pairs of zero modes with complex scaling dimensions. For $\gamma=0$ the singularities in the inverse symbols of operators M_n become strong enough to induce continuous spectrum of operators $\hat{N}_n(\sigma)$ and the picture of zero mode dominance has to be somewhat modified.^(1, 24)

One may also read the $\mathcal{O}(2-\gamma)$ contribution to the anomalous exponent ρ_{2n} from the $\mathcal{O}((2-\gamma) \ln L)$ term in the expansion of $\mathcal{F}_{2n,L}$ into powers of $2-\gamma$, similarly as in the ε -expansion for critical phenomena one obtains anomalous exponents from logarithmic divergences. In the latter case, the renormalization group which exponentiates the divergent logarithms provides an explanation why it is reasonable to extract information from badly divergent expansions. In our argument, the Mellin transform analysis played a similar role exponentiating the logarithms of L . One may show⁽¹³⁾ that there is an (inverse) renormalization group picture of the advection problem hidden behind the above analogy. The renormalization group for the passive scalar eliminates inductively the long-distance modes, unlike in critical phenomena where it is based on subsequent elimination of the short-distance degrees of freedom.

4. CONSERVED SCALING STRUCTURES

In view of the domination of the equal time correlators of the scalar by the scaling zero modes of operators M_n , it is important to understand the physical interpretation of such modes. It is, in fact, very simple:

zero modes are scaling structures preserved in mean by the flow.

Indeed, recall that $e^{-tM_n}(\mathbf{x}, \mathbf{x}_0) = P_n(t, \mathbf{x}; 0, \mathbf{x}_0)$ and it describes the probability that the differences of n Lagrangian trajectories starting at time 0 from points \mathbf{x}_0 are at time t equal to x_n . The mean value of a translationally

invariant function $f(\mathbf{x})$ of positions of n fluid particles at time t is then equal to

$$\langle f \rangle_{t, \mathbf{x}_0} \equiv \int f(\mathbf{x}) P_n(t, \mathbf{x}; 0, \mathbf{x}_0) d'\mathbf{x} = \int f(\mathbf{x}) e^{-tM_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \quad (4.1)$$

Differentiating the right hand side w.r.t. t , we obtain

$$\int f(\mathbf{x}) M_n e^{-tM_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} = \int M_n f(\mathbf{x}) e^{-tM_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \quad (4.2)$$

where we have integrated by parts twice. If f is a zero mode of M_n then the right hand side vanishes and, consequently, the mean (4.1) is constant and

$$\langle f \rangle_{t, \mathbf{x}_0} = f(\mathbf{x}_0) \quad (4.3)$$

In fact, the story is a little bit more complicated. The zero modes with $\text{Re } \sigma_i > -d_n + \gamma$ are true zero modes. However the ones with the real part of their dimension $\leq -d_n + \gamma$ are not. For them, $M_n f$ is a contact term supported at the origin. Such contact terms may give non-zero contributions to the right hand side of Eq. (4.2) or to the boundary terms in the integration by parts, depending on the interpretation. The zero modes with the scaling dimensions belonging to the strip (3.22) are true zero modes and hence they describe scaling structures of the flow conserved in mean. As was mentioned before, the translationally invariant zero modes that are square-integrable on S^{d_n-1} come in pairs (f_i, g_i) corresponding to scaling dimensions σ_i and $-d_n + \gamma - \bar{\sigma}_i$ (we may assume that $\text{Re } \sigma_i \geq -(d_n/2) + (\gamma/2)$). For γ close to 2 there are no zero modes square integrable on S^{d_n-1} with dimensions in the strip $-d_n + \gamma < \text{Re } \sigma < 0$. We expect this to hold for any $\gamma > 0$. In that situation f_i are the true zero modes and they have non-negative real parts of the scaling dimension whereas g_i are the false zero modes with real parts of dimension $\leq -d_n + \gamma$ and with $M_n g_i$ being contact terms. Our claim about the absence of zero modes in the strip $-d_n + \gamma < \text{Re } \sigma < 0$ may seem paradoxical if we recall that the multi-body structure of operators M_n assures that zero modes of M_{n-p} are also annihilated by M_n . Indeed,⁹ the (false) zero modes of M_{n-p} with $\text{Re } \sigma \leq -d_{n-p} + \gamma$ may lie in the strip $-(d_n/2) + (\gamma/2) < \text{Re } \sigma < 0$. However, the resulting zero modes of M_n are not in $L^2(S^{d_n-1})$ and do not contribute to

⁹ K. G. thanks E. Balkovsky, G. Falkovich and V. Lebedev for a discussion of this point.

the poles of the Green function $\hat{N}_n^{-1}(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ and hence to the right hand side of Eq. (3.21).

It should be stressed that the behavior (4.3) is atypical. For a general translationally invariant, scaling function with (say, positive) dimension σ and for any time $\tau > 0$,

$$\langle f \rangle_{t, \mathbf{x}_0} = \left(\frac{t}{\tau}\right)^{\sigma/\gamma} \int f(\mathbf{x}) e^{-\tau M_n} \left(\mathbf{x}, \left(\frac{\tau}{t}\right)^{1/\gamma} \mathbf{x}_0\right) d'\mathbf{x} \quad (4.4)$$

as it is easy to see with the use of the scaling property

$$e^{-\lambda^\gamma t M_n}(\lambda \mathbf{x}, \lambda \mathbf{x}_0) = \lambda^{-d_n} e^{-t M_n}(\mathbf{x}, \mathbf{x}_0) \quad (4.5)$$

It follows that, typically,

$$\langle f \rangle_{t, \mathbf{x}_0} \sim t^{\sigma/\gamma} \quad (4.6)$$

The behavior (4.6) characterizes a *super-diffusion* where the square distances between points grow faster than linearly in time. A slower behavior requires vanishing of $\int f(\mathbf{x}) e^{-\tau M_n}(\mathbf{x}, 0) d'\mathbf{x}$. Note that the exponent of the growth diverges when $\gamma \rightarrow 0$.

It is easy to understand the origin of the behavior (4.6). The stochastic process described by the probabilities $P_n(t, \mathbf{x}; 0, \mathbf{x}_0) = e^{-t M_n}(\mathbf{x}, \mathbf{x}_0)$ may be viewed as a diffusion with the diffusion coefficient proportional to the power $2 - \gamma$ of the distance between the particles. When particles separate they diffuse faster and faster which results in the super-diffusive behavior with mean distance square growing proportionally to $t^{2/\gamma}$. On the other hand, on small distances the diffusion is slow and particles which get close spend relatively long time together. Since $\langle f \rangle_{0, \mathbf{x}_0} = f(\mathbf{x}_0)$, the time t after which $\langle f \rangle_{t, \mathbf{x}_0}$ reaches, say, twice its original value behaves like $t^\gamma (f(\mathbf{x}_0)^{\gamma/\sigma})$, i.e., it goes slower to zero with the diminishing separation between the initial points than for the standard diffusion at $\gamma = 2$.

We have seen that the scaling zero modes of M_n correspond to *conserved collective modes* of the super-diffusion with the transition probabilities $P_n(t, \mathbf{x}; t_0, \mathbf{x}_0)$. Existence of such conserved modes is nothing exceptional. They are present already in the standard diffusion. For example,

$$\int [x_{12}^2 - x_{13}^2] e^{tA}(\mathbf{x}, \mathbf{x}_0) d\mathbf{x} = x_{0,12}^2 - x_{0,13}^2 \quad (4.7)$$

and is time independent, although $\int x_{ij}^2 e^{tA}(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}$ behaves like $\mathcal{O}(t)$. Another example is

$$\begin{aligned} & \int \left[x_{12}^2 x_{34}^2 - \frac{d}{2(d+2)} (x_{12}^4 + x_{34}^4) \right] e^{tA}(\mathbf{x}, \mathbf{x}_0) d\mathbf{x} \\ &= x_{0,12}^2 x_{0,34}^2 - \frac{d}{2(d+2)} (x_{0,12}^4 + x_{0,34}^4) \end{aligned} \quad (4.8)$$

Under symmetrization, the first conserved mode vanishes whereas the second one gives the zero mode of A whose $(2-\gamma)$ -perturbation dominates the 4-point function of the scalar for γ close to 2.

5. SOME PHYSICS: SHORT-DISTANCE BEHAVIOR OF FLUID PARTICLES

The zero mode dominance of the structure functions of the scalar is due to the appearance of such modes in the asymptotics of the Green functions $M_n^{-1}(\mathbf{x}, \mathbf{y})$. Indeed, with the use of the Mellin transform, one may write (in the reduced space):

$$\begin{aligned} M_n^{-1}(\mathbf{x}/L, \mathbf{y}) &= \int_{\text{Re } \sigma = -(d_n/2) + (\gamma/2)} \frac{d\sigma}{2\pi i} R_n(\mathbf{x}/L)^\sigma \hat{N}_n^{-1} \\ &\quad \times \left(\sigma - \frac{\gamma}{2}, \hat{\mathbf{x}}, \hat{\mathbf{y}} \right) R_n(\mathbf{y})^{-d_n + \gamma - \sigma} \end{aligned} \quad (5.1)$$

compare to Eq. (3.20). Pushing the integration contour more and more to the right and using Eq. (3.18) to control the residues of the poles, we obtain for $\mathbf{y} \neq 0$ the asymptotic large L expansion:

$$M_n^{-1}(\mathbf{x}/L, \mathbf{y}) = \sum_i L^{-\sigma_i} f_i(\mathbf{x}) \overline{g_i(\mathbf{y})} \quad (5.2)$$

with $\text{Re } \sigma_i > -(d_n/2) + (\gamma/2)$ or, as we expect, with $\text{Re } \sigma_i \geq 0$. Although it has a similar form to the eigenfunction expansion of an operator with discrete spectrum, it has little to do with the spectral decomposition of M_n^{-1} . Since M_n is a positive operator in $L^2(\mathbf{R}^{d_n})$ with continuous spectrum coinciding with the positive real line, the spectral decomposition of M_n^{-1} is a continuous integral involving the generalized eigen-functions of M_n . The scaling zero modes f_i or g_i of scaling dimensions σ_i and $-d_n + \gamma - \bar{\sigma}_i$,

respectively, are square-integrable on S^{d_n-1} but are not generalized eigenfunctions of M_n (except for $\sigma_i=0$). They are rather analogous to resonances in many-body problems with the plane of complex σ replacing that of complex energies and σ with real part equal to $-(d_n/2)$ corresponding to real energies.¹⁰ Note that due to the hermiticity and to the overall scaling of the Green function $M_n^{-1}(\mathbf{x}, \mathbf{y})$,

$$M_n^{-1}(\lambda\mathbf{x}, \lambda\mathbf{y}) = \lambda^{\gamma-d_n} M_n^{-1}(\mathbf{x}, \mathbf{y}) \tag{5.3}$$

the expansion (5.2) may be also rewritten as

$$M_n^{-1}(L\mathbf{x}, \mathbf{y}) = \sum_i L^{-d_n+\gamma-\bar{\sigma}_i} g_i(\mathbf{x}) \overline{f_i(\mathbf{y})} \tag{5.4}$$

so that the zero modes g_i with the scaling dimensions $-d_n+\gamma-\bar{\sigma}_i$ of real part less than $-(d_n/2)+(\gamma/2)$ (or even $\leq -d_n+\gamma$) dominate the large distance behavior of the Green function of M_n . Expansion (5.4) may be also obtained directly from Eq. (5.1) by pushing the σ -integration contour to the left.

It is not difficult to see directly that the functions f_i appearing in expansion (5.2) have to describe scaling structures conserved by the flow. Indeed,

$$\begin{aligned} & \int M_n^{-1}(\mathbf{x}/L, \mathbf{y}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \\ &= \sum_i L^{-\sigma_i} \overline{g_i(\mathbf{y})} \int f_i(\mathbf{x}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \end{aligned} \tag{5.5}$$

if we insert expansion (5.2) into the left hand side. But on the other hand,

$$\begin{aligned} & \partial_\tau \int M_n^{-1}(\mathbf{x}/L, \mathbf{y}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \\ &= \partial_\tau L^{d_n-\gamma} \int M_n^{-1}(\mathbf{x}, L\mathbf{y}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \\ &= -L^{d_n-\gamma} e^{-\tau M_n(\mathbf{x}_0, L\mathbf{y})} \end{aligned} \tag{5.6}$$

where we have used the scaling property (5.3). The (reduced space) heat kernel on the right hand side decays in L faster than any power for $\mathbf{y} \neq 0$.

¹⁰ This analogy is somewhat loose, since the poles in σ live in the first sheet, probably only on the real axis.

Comparing the latter expression to relation (5.5), we infer that $\partial_\tau \int f_i(\mathbf{x}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x}$ has to vanish and hence f_i , a function with scaling dimension σ_i , is conserved in mean by the Lagrangian flow. This gives another proof of the statement (4.3).

Since the Green function is given by the time integral of the heat kernel,

$$M_n^{-1}(\mathbf{x}, \mathbf{y}) = \int_0^\infty e^{-tM_n(\mathbf{x}, \mathbf{y})} dt \quad (5.7)$$

one may also expect to see the zero modes in the asymptotic behavior of the probabilities $P_n(t, \mathbf{x}; 0, \mathbf{x}_0) = e^{-tM_n(\mathbf{x}, \mathbf{x}_0)}$. Assume an asymptotic expansion

$$P_n(t, \mathbf{x}/L; 0, \mathbf{x}_0) = \sum_j L^{-\rho_j} \phi_j(\mathbf{x}) \overline{\psi_j(t, \mathbf{x}_0)}, \quad (5.8)$$

with $\text{Re } \rho_j \geq 0$, describing asymptotics of the probabilities that the Lagrangian trajectories will come at time t close together. We could expect that functions ϕ_j are again zero modes of M_n . To verify whether this is the case, consider the integral

$$\begin{aligned} & \int P_n(t, \mathbf{x}/L; 0, \mathbf{y}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \\ &= \sum_j L^{-\rho_j} \overline{\psi_j(t, \mathbf{y})} \int \phi_j(\mathbf{x}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \end{aligned} \quad (5.9)$$

The left hand side may be rewritten as

$$\begin{aligned} & L^{d_n} \int e^{-tM_n(\mathbf{x}, \mathbf{y})} e^{-\tau M_n(L\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \\ &= \int e^{-tM_n(\mathbf{x}, \mathbf{y})} e^{-L^{-\gamma} \tau M_n(\mathbf{x}, \mathbf{x}_0/L)} d'\mathbf{x} \\ &= e^{-(t+L^{-\gamma}\tau)M_n(\mathbf{x}_0/L, \mathbf{y})} = \sum_j L^{-\rho_j} \phi_j(\mathbf{x}_0) \overline{\psi_j(t+L^{-\gamma}\tau, \mathbf{y})} \end{aligned} \quad (5.10)$$

by changing variables $\mathbf{x} \mapsto L\mathbf{x}$ and using the scaling relations (4.5), the composition law of heat kernels and, finally, the expansion (5.8). But $\psi_j(t, \mathbf{y})$ should be smooth in t for $t \neq 0, \infty$. It then follows that

$$\int P_n(t, \mathbf{x}/L; 0, \mathbf{y}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} = \sum_{\substack{j \\ p=0, 1, \dots}} L^{-\rho_j - \gamma p} \frac{\tau^p}{p!} \overline{\partial_t^p \psi_j(t, \mathbf{y})} \phi_j(\mathbf{x}_0) \quad (5.11)$$

The right hand side becomes independent of τ only approximately if $\tau L^{-\gamma} \ll 1$. Comparing the asymptotic expansions (5.9) and (5.11), we infer that the scaling functions of (of scaling dimension ρ_j) are not necessarily preserved in mean by the Lagrangian flow. Instead, $\int \phi_j(\mathbf{x}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x}$ is a pure polynomial in τ . Note that the polynomial still grows slower than the super-diffusive growth $\tau^{\rho_j/\gamma}$ since its order p satisfies the relation $\rho_j = \rho_{j'} + \gamma p \geq \gamma p$ where $\rho_{j'}$ is a scaling dimension of some other $\phi_{j'}$. Hence functions ϕ_j describe *slow collective modes* of the super-diffusion.

It is not difficult to see how the slow modes are related to the zero modes of operators M_n . Differentiating Eq. (5.8) over t we infer that

$$\begin{aligned} -\sum_j L^{-\rho_j} \phi_j(\mathbf{x}) \overline{\partial_t \psi_j(t, \mathbf{x}_0)} &= \sum_j L^{-\rho_j} \phi_j(\mathbf{x}) M_n \overline{\psi_j(t, \mathbf{x}_0)} \\ &= \sum_j L^{-\rho_j + \gamma} M_n \phi_j(\mathbf{x}) \overline{\psi_j(t, \mathbf{x}_0)} \end{aligned} \quad (5.12)$$

It follows that if the function ϕ_j appears in the expansion (5.8) then also $M_n \phi_j$ does. Since the scaling dimension of $M_n \phi_j$ is $\rho_j - \gamma$ and only dimensions with real part positive may appear, subsequent application of M_n to a ϕ_j must produce a homogeneous zero mode of M_n after a finite number of steps. Hence functions ϕ_j must be organized into towers of descendants $\phi_{i,p}$ based at zero modes $f_i \equiv \phi_{i,0}$ of M_n and satisfying the chain of equations

$$M_n \phi_{i,p} = \phi_{i,p-1}, \quad p = 1, \dots \quad (5.13)$$

The scaling dimension of $\phi_{i,p}$ is $(\sigma_i + \gamma p)$. Since $M_n^{p+1} \phi_{i,p} = 0$, it follows that the $(p+1)$ th time derivative of

$$\int \phi_{i,p}(\mathbf{x}) e^{-\tau M_n(\mathbf{x}, \mathbf{x}_0)} d'\mathbf{x} \quad (5.14)$$

vanishes so that the above integral is a polynomial in τ of degree p , in accordance with the previous reasoning. Note that Eq. (5.12) implies that the functions $\psi_{i,p}$ corresponding to $\phi_{i,p}$ satisfy

$$\psi_{i,p} = -\partial_t \psi_{i,p-1} = M_n \psi_{i,p-1} \quad (5.15)$$

Summarizing: the asymptotics of probabilities of the Lagrangian trajectories to get close together is dominated by the towers of slow collective modes of the super-diffusion:

$$P_n(t, \mathbf{x}/L; 0, \mathbf{x}_0) = \sum_{p=0,1,\dots}^i L^{-\sigma_i - \gamma p} \phi_{i,p}(\mathbf{x}) \overline{\psi_{i,p}(t, \mathbf{x}_0)} \quad (5.16)$$

Since $P_n(t, \mathbf{x}; 0, \mathbf{x}_0) = P_n(t, \mathbf{x}_0; 0, \mathbf{x})$, expansion (5.16) may be also rewritten as

$$P_n(t, \mathbf{x}; 0, \mathbf{x}_0/L) = \sum_{p=0,1,\dots}^i L^{-\sigma_i - \gamma p} \overline{\psi_{i,p}(t, \mathbf{x})} \phi_{i,p}(\mathbf{x}_0) \quad (5.17)$$

giving the asymptotics of the probabilities of the Lagrangian trajectories starting very close. The leading term on the right hand side is equal to $\overline{\psi_{0,0}(t, \mathbf{x})} = e^{-tM_n(\mathbf{x}, 0)}$ and it corresponds to the constant zero mode $\phi_{0,0} = f_0 = 1$.

The above results have an important physical significance for the dynamics of the scalar. Recall the expression (4.4) for the time-dependence of the mean value $\langle f \rangle_{t, \mathbf{x}_0}$ of a function f of scaling dimension σ . Inserting the relation (5.17) to the right hand side of Eq. (4.4) we obtain the asymptotic expansion of $\langle f \rangle_{t, \mathbf{x}_0}$ for large t :

$$\langle f \rangle_{t, \mathbf{x}_0} = \sum_{p=0,1,\dots}^i \left(\frac{t}{\tau} \right)^{((\sigma - \sigma_i)/\gamma) - p} \phi_{i,p}(\mathbf{x}_0) \int f(\mathbf{x}) \overline{\psi_{i,p}(\tau, \mathbf{x})} d'\mathbf{x} \quad (5.18)$$

The leading term corresponds to the constant zero mode. This term dominates for large t if $\int f(\mathbf{x}) e^{-\tau M_n(\mathbf{x}, 0)} d'\mathbf{x} \neq 0$. However if $f = \phi_{l,q}$ then, as we have seen above, $\langle \phi_{l,q} \rangle_{t, \mathbf{x}_0}$ is a polynomial of order q in t . Consequently, $\int \phi_{l,q}(\mathbf{x}) \overline{\psi_{i,p}(\tau, \mathbf{x})} d'\mathbf{x}$ has to vanish unless $((\sigma_l - \sigma_i)/\gamma) + q - p$ is an integer between 0 and q . Hence $\langle f \rangle_{t, \mathbf{x}_0}$ for f equal to a slow mode $\phi_{l,q}$ with a positive scaling dimension is dominated by subleading terms on the right hand side of Eq. (5.18).

To see the lower order terms¹¹ for a generic scaling function f for which $\langle f \rangle_{t, \mathbf{x}_0} \sim t^{\sigma/\gamma}$, it is enough to compare the mean values $\langle f \rangle_{t, \mathbf{x}_0}$ for two different \mathbf{x}_0 . For example, subtracting $\langle f \rangle_{t, \mathbf{x}_0}$ for two values of $x_{0,1}$

¹¹ K. G. thanks M. Vergassola for the discussion of this point.

gets rid of the contribution of the constant mode. Denote by $\delta_{y_{0,m}, y'_{0,m}}$ the operator which performs the subtraction on functions h of $x_{0,m}$:

$$\delta_{y_{0,m}, y'_{0,m}} h = h(y_{0,m}) - h(y'_{0,m}) \tag{5.19}$$

Subtracting subsequently at two different values of $x_{0,m}$ for $m = 1, \dots, n - 1$, and setting $\prod_{m=1}^{n-1} \delta_{y_{0,m}, y'_{0,m}} \equiv \delta_{y, y'}$, we obtain

$$\delta_{y, y'} \langle f \rangle_{t, \cdot} = \sum'_{i, p=0, 1, \dots} \left(\frac{t}{\tau}\right)^{((\sigma - \sigma_i)/\gamma) - p} (\delta_{y, y'} \phi_{i, p}) \int f(\mathbf{x}) \overline{\psi_{i, p}(\tau, \mathbf{x})} d' \mathbf{x} \tag{5.20}$$

where the primed sum omits the contributions of the slow modes which do not depend on all variables. Hence the non-constant slow modes dominate the relative motion of groups of Lagrangian trajectories starting from different initial configurations. In particular, the relative motion is slower than the super-diffusive spread of the trajectories. This supports the interpretation of the slow modes as *resonance-type* objects in the motion of Lagrangian trajectories. The slow modes depending on less variables correspond to resonances in fewer-particle channels which drop out under the subtractions.

6. SOME MATHEMATICS: STRUCTURE OF THE MULTI-BODY OPERATORS M_n

To understand analytically the origin of the asymptotic expansion (5.16), let us examine more closely operators M_n . We shall work in the reduced space $\mathcal{H} \equiv L^2(\mathbf{R}^{d_n})$. M_n is a positive, unbounded, self-adjoint operator in \mathcal{H} . Let

$$(\mathcal{U}_s f)(\mathbf{x}) = e^{s d_n / 2} f(e^s \mathbf{x}) \tag{6.1}$$

Operators \mathcal{U}_s form a unitary version of the 1-parameter group of dilations in \mathcal{H} with the self-adjoint operator D_n of Eq. (3.14) as its generator:

$$\mathcal{U}_s = e^{is D_n} \tag{6.2}$$

\mathcal{U}_s preserve the domain of M_n and

$$\mathcal{U}_s M_n \mathcal{U}_s^{-1} = e^{-\gamma s} M_n \tag{6.3}$$

Denote by X_n the natural logarithm of operator M_n : $X_n = \ln M_n$. X_n is an unbounded self-adjoint operator on \mathcal{H} with the domain invariant under \mathcal{U}_s

and the whole real line as the spectrum. The relation (6.3) is equivalent to

$$\mathcal{U}_s X_n \mathcal{U}_s^{-1} = X - \gamma s \quad (6.4)$$

i.e., to a strong form of the canonical commutation relation

$$[D_n, X_n] = i\gamma \quad (6.5)$$

Since under the Mellin transform (3.15) D_n becomes the multiplication operator by $(1/i)(\sigma + (d_n/2))$, X_n must be unitarily equivalent to $\gamma \partial_\sigma$ by virtue of the von Neumann Theorem on representations of the canonical commutation relations. More exactly, there exists a one-parameter family $\hat{U}_n(\sigma)$, $\text{Re } \sigma = -d_n/2$, of unitary operators in $L^2(S^{d_n-1})$, unique modulo a right multiplication by a σ -independent unitary operator, such that

$$(X_n f)^\wedge(\sigma, \cdot) = \gamma \hat{U}_n(\sigma) \partial_\sigma \hat{U}_n^{-1}(\sigma) \hat{f}(\sigma, \cdot) \quad (6.6)$$

for $\text{Re } \sigma = -d_n/2$. Equivalently,

$$(M_n f)^\wedge(\sigma, \cdot) = \hat{U}_n(\sigma) e^{\gamma \partial_\sigma} \hat{U}_n^{-1}(\sigma) \hat{f}(\sigma, \cdot) \quad (6.7)$$

or, noting that the operator ∂_σ corresponds to the multiplication by $-\ln R_n$ in the language of the original functions,

$$M_n = U_n R_n^{-\gamma} U_n^{-1} \quad (6.8)$$

where

$$(U_n f)^\wedge(\sigma, \cdot) = \hat{U}_n(\sigma) \hat{f}(\sigma, \cdot) \quad (6.9)$$

This is the promised structural result about operators M_n . For the heat kernels, we obtain

$$e^{-tM_n} = U_n e^{-tR_n^{-\gamma}} U_n^{-1} \quad (6.10)$$

What is the relation of the expression (6.8) to the representation

$$M_n = R_n^{-\gamma/2} N_n R_n^{-\gamma/2} \quad (6.11)$$

used before, with N_n an operator commuting with D_n ? The comparison of the two equations gives

$$N_n = R_n^{\gamma/2} U_n R_n^{-\gamma} U_n^{-1} R_n^{\gamma/2} \quad \text{or} \quad N_n^{-1} = R_n^{-\gamma/2} U_n R_n^\gamma U_n^{-1} R_n^{-\gamma/2} \quad (6.12)$$

Let us suppose that the family of operators $\hat{U}_n(\sigma)$ has a meromorphic continuation to the complex plane of σ with no poles in the strip

$$-\frac{d_n}{2} \leq \text{Re } \sigma \leq -\frac{d_n}{2} + \frac{\gamma}{2} \quad (6.13)$$

(this will prove consistent with our zero mode analysis). Then $R_n^{-\gamma/2} U_n R_n^{\gamma/2}$ becomes under the Mellin transform the operator

$$e^{(\gamma/2)\partial_\sigma} \hat{U}_n(\sigma) e^{-(\gamma/2)\partial_\sigma} = \hat{U}_n\left(\sigma + \frac{\gamma}{2}\right)$$

The unitarity of \hat{U}_n for $\text{Re } \sigma = -d_n/2$ implies that

$$\hat{U}_n(\sigma) \hat{U}_n(-d_n - \bar{\sigma})^* = \hat{U}_n(-d_n - \bar{\sigma})^* \hat{U}_n(\sigma) = 1 \quad (6.14)$$

and that $\hat{U}_n^{-1}(\sigma)$ also possesses a meromorphic continuation. Operator $R_n^{\gamma/2} U_n^{-1} R_n^{-\gamma/2}$ becomes $\hat{U}_n^{-1}(\sigma - \gamma/2)$ under the Mellin transform and Eq. (6.12) gives rise to the relation

$$\hat{N}_n^{-1}\left(\sigma - \frac{\gamma}{2}\right) = \hat{U}_n(\sigma) \hat{U}_n^{-1}(\sigma - \gamma) \quad (6.15)$$

If $\hat{N}_n^{-1}(\sigma - \gamma/2)$ has a pole, see Eq. (3.18), then the simplest possibility is that either $\hat{U}_n(\sigma_i - \gamma)$ is regular and then

$$\hat{U}_n(\sigma) = \frac{1}{\sigma - \sigma_i} |f_i\rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + \mathcal{O}(1) \quad (6.16)$$

when $\sigma \rightarrow \sigma_i$ or $\hat{U}_n(\sigma_i)$ is regular and

$$\hat{U}_n(\sigma_i) = \frac{1}{\sigma - \sigma_i} |f_i\rangle \langle g_i| \hat{U}_n(\sigma - \gamma) + \mathcal{O}(\sigma - \sigma_i) \quad (6.17)$$

with $\langle g_i| \hat{U}_n(\sigma - \gamma) = \mathcal{O}(\sigma - \sigma_i)$. In the last case, multiplying by $\hat{U}_n(-d_n - \bar{\sigma}_i)^*$ from the left and by $\hat{U}_n(-d_n - \bar{\sigma} + \gamma)^* = \hat{U}_n^{-1}(\sigma - \gamma)$ from the right hand side and taking adjoints, we obtain

$$\hat{U}_n(-d_n - \bar{\sigma} + \gamma) = \frac{1}{\bar{\sigma} - \bar{\sigma}_i} |g_i\rangle \langle f_i| \hat{U}_n(-d_n - \bar{\sigma}_i) + \mathcal{O}(1) \quad (6.18)$$

i.e., relation (6.16) for σ_i replaced by $-d_n + \gamma - \bar{\sigma}_i$ and corresponding to the twin zero mode g_i of f_i . We expect the behavior (6.16) if f_i is less singular at the origin than g_i and the behavior (6.17) in the opposite case (in Appendix A, this is established for \hat{U}_2). Rewrite Eq. (6.15) as

$$\hat{U}_n(\sigma + \gamma p) = \hat{N}_n^{-1}(\sigma + \gamma(p - \frac{1}{2})) \hat{U}_n(\sigma + \gamma(p - 1)) \quad (6.19)$$

Assume that $\hat{N}_n^{-1}(\sigma_i + \gamma(p - \frac{1}{2}))$ is regular for $p = 1, 2, \dots$, (i.e., that there are no zero modes of M_n (square-integrable on S^{d_n-1}) with scaling dimensions differing by multiplicity of γ . This should hold for generic γ . From Eq. (6.19) for $p = 1$ and from relation (6.16) we infer that

$$\begin{aligned} \hat{U}_n(\sigma + \gamma) &= \frac{1}{\sigma - \sigma_i} \hat{N}_n^{-1}\left(\sigma + \frac{\gamma}{2}\right) |f_i\rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + \mathcal{O}(1) \\ &\equiv \frac{1}{\sigma - \sigma_i} |\phi_{i,1}\rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + \mathcal{O}(1) \end{aligned} \quad (6.20)$$

for $\sigma \rightarrow \sigma_i$. By induction on p , it follows then that

$$\hat{U}_n(\sigma + \gamma p) = \frac{1}{\sigma - \sigma_i} |\phi_{i,p}\rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + \mathcal{O}(1) \quad (6.21)$$

where

$$\phi_{i,p} = \hat{N}_n^{-1}(\sigma_i + \gamma(p - \frac{1}{2})) \phi_{i,p-1} \quad (6.22)$$

Note that Eqs. (6.22) may be rewritten as the chain of relations (5.13) for the scaling functions $\phi_{i,p}(\mathbf{x}) \equiv R_n^{\sigma_i + \gamma p} \phi_{i,p}(\hat{\mathbf{x}})$ where $\phi_{i,0} \equiv f_i$ is the zero mode of M_n of scaling dimension σ_i . By virtue of the assumption that there are no zero modes of M_n of scaling dimension $\sigma_i + \gamma p$, the tower of descendants $\phi_{i,p}$ is uniquely determined¹² for each zero mode f_i .

For f a test function vanishing near the origin, Eq. (6.10) may be rewritten as

$$\begin{aligned} (e^{-tM_n f})^\wedge(\sigma, \hat{\mathbf{x}}) &= \int d\hat{\mathbf{y}} \hat{U}_n(\sigma; \hat{\mathbf{x}}, \hat{\mathbf{y}}) \int_0^\infty d\lambda \lambda^{-\sigma-1} e^{-t\lambda} \\ &\quad \times \int_{\text{Re } \sigma' = -d_n/2} \frac{d\sigma'}{2\pi i} \lambda^{\sigma'} (\hat{U}_n^{-1} \hat{f})(\sigma', \hat{\mathbf{y}}) \end{aligned} \quad (6.23)$$

¹² For non-generic γ the situation may be slightly more complicated with mixing of different towers.

After shifting the σ' -integration contour infinitesimally to the left we may perform the λ -integral. The inverse Mellin transform of the resulting expression gives

$$\begin{aligned}
 & (e^{-tM_n f})(\hat{\mathbf{x}}/L) \\
 &= \frac{1}{\gamma} \int_{\text{Re } \sigma = -d_n/2} \frac{d\sigma}{2\pi i} L^{-\sigma} \int d\hat{\mathbf{y}} \hat{U}_n(\sigma; \hat{\mathbf{x}}, \hat{\mathbf{y}}) \\
 & \quad \times \int_{\text{Re } \sigma' = -d_n/2 - 0} \frac{d\sigma'}{2\pi i} t^{(\sigma' - \sigma)/\gamma} \Gamma\left(\frac{\sigma - \sigma'}{\gamma}\right) (\hat{U}_n^{-1} \hat{f})(\sigma', \hat{\mathbf{y}}) \quad (6.24)
 \end{aligned}$$

By moving the σ -integration contour further and further to the right,¹³ we obtain from Eq. (6.24) the asymptotic expansion

$$(e^{-tM_n f})(\hat{\mathbf{x}}/L) = \sum_{p=0, 1, \dots} L^{-\sigma_i - \gamma p} \phi_{i,p}(\hat{\mathbf{x}}) \overline{\psi_{i,p}} \quad (6.25)$$

where the sum is over scaling dimensions of zero modes of M_n satisfying $\text{Re } \sigma_i > -d_n/2$ and where

$$\begin{aligned}
 \overline{\psi_{i,p}} &= -\frac{1}{\gamma} \int d\hat{\mathbf{y}} \overline{(\hat{U}_n(\sigma_i - \gamma) * g_i)(\hat{\mathbf{y}})} \\
 & \quad \times \int_{\text{Re } \sigma' = -d_n/2} \frac{d\sigma'}{2\pi i} t^{((\sigma' - \sigma_i)/\gamma) - p} \Gamma\left(\frac{\sigma_i - \sigma'}{\gamma} + p\right) \\
 & \quad \times (\hat{U}_n^{-1} \hat{f})(\sigma', \hat{\mathbf{y}}) \quad (6.26)
 \end{aligned}$$

Eq. (6.25) is an integrated version of expansion (5.16) (at least for γ close to 2, there are no scaling zero modes square-integrable on S^{d_n-1} with $-d_n < \text{Re } \sigma_i < 0$ and, as mentioned before, we expect this to hold for all positive γ).

7. LAGRANGIAN FLOW

The study of the statistics of lagrangian trajectories has a long history, see e.g., refs. 21 and 23. There are, however, some subtle but important points which, to our knowledge, have not been stressed before and which

¹³ Note that the poles of the Γ -function do not contribute.

the above discussion of the motion of fluid particles allows to elucidate. When stating that, for $m, \kappa = 0$, $P_n(t, \mathbf{x}; 0, \mathbf{x}_0)$ is the joint p.d.f. of the differences of the endpoints of n Lagrangian trajectories, we have silently assumed that such trajectories, or at least their differences, make sense as random processes defining the Lagrangian flow on the probability space of v 's. A straightforward consequence of such an assumption is the relation

$$P_n(t, \mathbf{x}; 0, \mathbf{x}_0)|_{x_{0,k}=x_{0,k+1}=\dots=x_{0,n}} = P_n(t, \mathbf{x}'; 0, \mathbf{x}'_0) \prod_{i=k}^n \delta(x_{in}) \quad (7.1)$$

where $\mathbf{x}' = (x_1, \dots, x_k)$ and similarly for \mathbf{x}'_0 . Eq. (7.1) expresses the elementary property that the joint p.d.f. of coinciding random variables is concentrated on the diagonal. In particular, $P_n(t, \mathbf{x}; 0, 0)$ should be proportional to the d_n -dimensional delta-function. But the heat kernels $e^{-tM_n(\mathbf{x}, \mathbf{x}_0)}$ do not have this property at least for γ close to 2 and, expectedly, for all $\gamma > 0$. Instead they are regular when $\mathbf{x}_0 \rightarrow 0$. How exactly $P_n(t, \mathbf{x}; 0, \mathbf{x}_0/L)$ fails to become the delta-function when L goes to infinity is described by the asymptotics (5.17) dominated by the slow collective modes of the stochastic evolution of the Lagrangian trajectories. Hence, even if all joint p.d.f.'s P_n of the differences of Lagrangian trajectories make sense as given by the $m, \kappa = 0$ heat kernels $e^{-tM_n(\mathbf{x}, \mathbf{x}_0)}$, the differences of Lagrangian trajectories do not exist as random processes for $\gamma > 0$. Note that for κ positive we should not expect the behavior (7.1) since the Brownian motions starting from $x_{0,i}$ are different for different i 's even if they wiggle around the same Lagrangian trajectory. The system behaves as if the wiggles were present even for $\kappa = 0$ (see more on that below).

The $\gamma = 0$ and $m, \kappa = 0$ case will be analyzed in Sects. 8 and 9 and was previously considered in refs. 19, 25 and 26 to ref. 12. The p.d.f. $P_2(t, x; 0, x_0) = e^{-tM_2(x, x_0)}$ of the difference $x_{12}(t) \equiv x(t)$ of two Lagrangian trajectories may be easily computed in this case and the result is the log-normal distribution^(6, 8, 19)

$$P_2(t, x; 0, x_0) = \frac{r^{-d}}{\sqrt{4\pi Dt}} e^{-(1/(4Dt))(\ln(r/r_0) - tDd)^2} k_{D(d+1)/(d-1)t}(\hat{x}, \hat{x}_0) \quad (7.2)$$

where $r = |x|$, $\hat{x} = x/r$ and similarly for r_0, \hat{x}_0 . $k_{D(d+1)/(d-1)t}(\hat{x}, \hat{x}_0)$ denotes the heat kernel on the unit sphere in d dimensions and it drops out in the rotationally invariant sector. The most important consequence of Eq. (7.2) is that $1/t \ln(r/r_0)$ is a Gaussian variable with covariance $2D/t$ tending to zero at large times and with mean Dd . The mean gives the *Lyapunov exponent*

i.e., the rate of exponential growth in time of the distance r between the Lagrangian trajectories. Note that

$$\int r^\sigma P_2(t, x; 0, x_0) dx = r_0^\sigma e^{D\sigma(d+\sigma)t} \tag{7.3}$$

which should be contrasted with the super-diffusive behavior for $\gamma > 0$ described by Eq. (4.6). More generally,

$$\begin{aligned} & \int f(x) P_2(t, x; 0, x_0) dx \\ &= \int f(e^{ur_0}\hat{x}) \frac{1}{\sqrt{4\pi Dt}} e^{-(1/(4Dt))(u-Dd)^2} k_{D((d+1)/(d-1))t}(\hat{x}, \hat{x}_0) du d\hat{x} \end{aligned} \tag{7.4}$$

For any test function f and for fixed t , the right hand side tends to $f(0)$ when $r_0 \rightarrow 0$, in accordance with the relation (7.1) and unlike for $\gamma > 0$. As it is easy to see from the above integral (or from Eq. (7.2)), the concentration of the p.d.f. $P_2(t, x; 0, x_0)$ within $r \lesssim \eta$ is visible if $\eta \gg e^{tDd}r_0$. For later use, note that for a small but non-zero r_0 and for a rotationally invariant test function f ,

$$\begin{aligned} & \int f(x) P_2(t, x; 0, x_0) dx \\ &= \int f(e^{ur_0}) \frac{\sqrt{t}}{\sqrt{4\pi D}} e^{-(t/(4D))(u-Dd)^2} du \xrightarrow{t \rightarrow \infty} 0 \end{aligned} \tag{7.5}$$

The approach of refs. 6 and 8 was based on the observation that at $\gamma = 0$ the 2-point function of the velocity differences becomes

$$\begin{aligned} & \langle (v^\alpha(t_1, x_1) - v^\alpha(t_1, x_2))(v^\beta(t_2, x'_1) - v^\beta(t_2, x'_2)) \rangle \\ &= 2 \frac{D}{d-1} \delta(t_{12}) [(d+1) \delta^{\alpha\beta} x_{12} \cdot x'_{12} - x_{12}^\alpha x'^\beta_{12} - x'^\alpha_{12} x^\beta_{12}] \end{aligned} \tag{7.6}$$

In particular, the first derivatives of v have space-independent correlations. In other words, we may set

$$v(t, x_1) - v(t, x_2) = X(t) x_{12} \quad \text{or} \quad \partial_y v^\alpha(t, x) = X^{\alpha\gamma}(t) \tag{7.7}$$

where $X^{\alpha\beta}(t)$ is a Gaussian process with values in traceless matrices with mean zero and the 2-point function

$$\langle X^{\gamma\alpha}(t) X^{\delta\beta}(s) \rangle = 2 \frac{D}{d-1} \delta(t-s) [(d+1) \delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma}] \quad (7.8)$$

obtained by differentiating twice the right hand side of (7.6). It is easy to check directly that the above covariance is positive and that it is invariant under the adjoint action of $O(d)$, i.e., that X and kXk^{-1} have the same covariance for $k \in O(d)$. Eqs. (7.7) are equalities between Gaussian processes. Physically, they mean that for $m=0$ and $\gamma=0$ the velocity flow acts as a uniform, volume preserving strain and rotation, as far as the relative motions of fluid particles are concerned. The difference of two Lagrangian trajectories $x_{12}(t) \equiv x(t)$ should satisfy the linear (stochastic) ODE

$$dx = X(t) x dt, \quad x(0) = x_0 \quad (7.9)$$

with a solution given formally by

$$x(t) = g_{t, t_0} x_0 \quad (7.10)$$

where g_{t, t_0} is the time-ordered exponential of an integral of independent matrices,

$$g_{t, t_0} = \mathcal{T} e^{\int_{t_0}^t X(s) ds} \quad (7.11)$$

of the type similar to the ones that appears in the theorems on products of independent equally distributed matrices⁽¹¹⁾ or in the one-dimensional Anderson localization.⁽¹⁶⁾ The point is that g_{t, t_0} may be defined as a random Markov process (a diffusion) with values in $SL(d)$. It has three basic properties:

1. g_{t_2, t_1} and $g_{t_2+\tau, t_1+\tau}$ have the same distribution,
2. $g_{t_2, t_1} g_{t_1, t_0} = g_{t_2, t_0}$ a.e.,
3. g_{t, t_0} is independent of g_{t', t_0} if $(t_0, t) \cap (t_0, t') = \emptyset$.

To define such a process, it is enough to give the (transition) probability distributions $p_{t-t_0}(g) dg$ of g_{t, t_0} (dg denotes the Haar measure on $SL(d)$) satisfying the composition law:

$$\int p_t(g) p_s(g^{-1}h) dg = p_{t+s}(h) \quad (7.12)$$

The $SO(d)$ -invariance of the Lie-algebra-valued process X imposes also the relation

$$p_t(kgk^{-1}) = p_t(g) \tag{7.13}$$

In Sect. 8 we identify p_t with the heat kernel of a certain operator on $SL(d)$.

The net outcome of that analysis is that for $\gamma = 0$, unlike for $\gamma > 0$, the differences of Lagrangian trajectories $x_{ij}(t) = g_{t, t_0} x_{0, ij}$ are well defined random variables. In particular, the knowledge of p_t is all that is needed to compute the joint p.d.f.'s of $x_{ij}(t)$:

$$\int P_n(t, \mathbf{x}; 0, \mathbf{x}_0) f(\mathbf{x}) d'\mathbf{x} = \int_{SL(d)} p_t(g) f(g\mathbf{x}_0) dg \tag{7.14}$$

for translationally invariant f . In fact, the above integrals uniquely determine p_t .

One of the consequences of the relation (7.14), closely related to the property (7.1), is that, for $\gamma = 0$, the stochastic evolution of the scalar T defined by the $m, \kappa = 0$ flow preserves the Gibbs measure formally given as $e^{-\beta \int T^2 DT}$ /normalization. Indeed, the $2n$ -point function of the scalar in this measure is

$$\mathcal{F}_n^{\text{Gibbs}}(\mathbf{x}) = \frac{(2n)!}{2^{2n} n! \beta^n} \mathcal{S} \delta(x_{12}) \delta(x_{34}) \cdots \delta(x_{2n-1, 2n}) \tag{7.15}$$

(the odd functions vanish). But Eq. (7.14) implies the relation

$$\int P_{2n}(t, \mathbf{x}; 0, \mathbf{x}_0) \mathcal{F}_n^{\text{Gibbs}}(\mathbf{x}_0) d'\mathbf{x}_0 = \mathcal{F}_n^{\text{Gibbs}}(\mathbf{x}) \tag{7.16}$$

i.e., the time invariance of the Gibbs measure correlations for $\gamma = 0$. This should be contrasted with the behavior for the $\gamma > 0$ case where the flux of the scalar energy towards high wavenumbers destroys the invariance of the Gibbs measure, see ref. 14. For $\gamma = 0$, the invariant Gibbs measure is nevertheless unstable under perturbations, as follows from relation (7.5). It has also little to do with the $\kappa \rightarrow 0$ limit of the stationary state of the scalar obtained in the presence of large scale forcing. The latter will be constructed in Sect. 9.

The mathematics of the difference between the $\gamma = 0$ and $\gamma > 0$ cases is simple. Eq. (2.8) requires that $v(t, x)$ be Lipschitz in x for the uniqueness

of solutions.¹⁴ But the Gaussian v -measure with 2-point function (3.4) lives on v 's which are Hölder in x with exponent $(2 - \gamma)/2$ (modulo logarithmic corrections) but not Lipschitz, except for $\gamma = 0$ where the velocity differences become smooth, as we have seen above. Hence, one should not expect uniqueness of Lagrangian trajectories even if the probabilistic description of them may be maintained but with violation of the property (7.1). Physically,¹⁵ the velocity covariance should be smoothed on the dissipative scale η due to viscous effects so that it behaves as $\sim D\eta^{-\gamma}r^2$ for $r \ll \eta$, i.e., like the $\gamma = 0$ covariance with D increased to $D\eta^{-\gamma}$. The Lagrangian trajectories diverge now exponentially in time as long as their distance is $\ll \eta$. Note, however, that for arbitrary small but fixed $r_0 = |x_0|$ one never sees concentration of the p.d.f. $P_2(t, x; 0, x_0)$ on scales smaller than η if $\eta \lesssim e^{o(\eta^{-\gamma})}r_0$, i.e., for η sufficiently small. This explains in more physical terms why relation (7.1) fails when $\eta \rightarrow 0$ for $\gamma > 0$. The exponential divergence of trajectories closer than η makes it impossible to maintain the concept of (differences of) individual trajectories in the inviscid limit $\eta \rightarrow 0$. Instead, we should talk about the v -dependent p.d.f. $P_n(t, \mathbf{x}; 0, \mathbf{x}_0|v)$ whose average over the velocity ensemble reproduces $P_n(t, \mathbf{x}; 0, \mathbf{x}_0)$. It is worth noting that for positive diffusivity κ , when the deterministic equation (2.8) should be replaced by the stochastic ODE (2.9), although the problems with the non-uniqueness of the solutions persist, there exists a rigorous probabilistic treatment¹⁶ allowing to define uniquely the transition probabilities $P_n(t, \mathbf{x}; 0, \mathbf{x}_0|v)$ for Hölder continuous velocities.⁽²⁷⁾ Our analysis calls for an extension of such a treatment to the $\kappa = 0$ case.

8. ADVECTION BY SMOOTH VELOCITIES AND HARMONIC ANALYSIS

When $\gamma = 0$ and $m, \kappa = 0$, the Kraichnan model becomes exactly solvable as we will show now. That simplifications occur in this case has been noted before, see refs. 24 and 25 to ref. 10. Our analysis is based on some observations by Shraiman and Siggia.^(24, 26) As was noted in ref. 24 and in ref. 26, for $\gamma = 0$ the model has extra symmetries. The operators M_n can be expressed in terms of the quadratic Casimir operators corresponding to an action of the groups $SL(d)$ and $SO(d)$ on the correlation functions. Let us explain what this means.

The group $SL(d)$ of real matrices of determinant 1 acts on functions f on \mathbf{R}^d on the left by $(L_g f)(x) = f(g^{-1}x)$. The infinitesimal form of this

¹⁴ Recall the existence of two solutions with vanishing initial condition: $x = ((\gamma/2)t)^{2/\gamma}$ and $x = 0$, for the equation $\dot{x} = x^{(2-\gamma)/2}$.

¹⁵ K. G. thanks G. Falkovich for a discussion of this point.

¹⁶ We thank G. Eyink for pointing this out to us.

action is given by $(d/dt)|_{t=0} L_{e^A} f = A^{\beta\alpha} H_{\alpha\beta} f$ where A is a traceless matrix (i.e., in the Lie algebra of $SL(d)$) and the generators $H_{\alpha\beta}$ are

$$H_{\alpha\beta} = -x^\alpha \partial_{x^\beta} + \frac{1}{d} \delta_{\alpha\beta} x^\gamma \partial_{x^\gamma} \tag{8.1}$$

Similarity, on functions of n \mathbf{R}^d variables $(x_1, \dots, x_n) = \mathbf{x}$, we have the (diagonal) action

$$(L_g f)(\mathbf{x}) = f(g^{-1}\mathbf{x}) \tag{8.2}$$

with generators $H_{\alpha\beta} = \sum_i (-x_i^\alpha \partial_{x_i^\beta} + (1/d) \delta_{\alpha\beta} x_i^\gamma \partial_{x_i^\gamma})$. The quadratic Casimir of $SL(d)$ is in terms of these generators

$$H^2 = \sum_{\alpha, \beta} H_{\alpha\beta} H_{\beta\alpha} \tag{8.3}$$

The generators of the action of the $SO(d)$ subgroup are $J_{\alpha\beta} = H_{\alpha\beta} - h_{\beta\alpha}$ and the corresponding quadratic Casimir is

$$J^2 = -\frac{1}{2} \sum_{\alpha, \beta} J_{\alpha\beta}^2 \tag{8.4}$$

The observation of Shraiman and Siggia was that when $\gamma, m=0$ in the velocity covariance $d^{\alpha\beta}$ (3.3), the operator $M_n = \sum_{i < j} d^{\alpha\beta} (x_i - x_j) \partial_{x_i^\alpha} \partial_{x_j^\beta}$ becomes

$$M_n = \frac{D}{d-1} [(d+1) J^2 - dH^2] \tag{8.5}$$

By definition of the action (8.2), the same formula holds also when we express M_n in terms of the $n-1$ difference variables. In particular for M_2 , the Casimirs H^2 and J^2 correspond to the action of $SL(d)$ and $SO(d)$ on functions $f(x)$ of the difference variable $x \equiv x_{12}$. In this case, we may diagonalize H^2 by the Mellin transform $f(x) \rightarrow \hat{f}(\sigma, \hat{x}) = \int_0^\infty r^{-\sigma-1} f(r\hat{x}) dr$:

$$(H^2 f)^\wedge(\sigma, \hat{x}) = \frac{d-1}{d} \sigma(\sigma+d) \hat{f}(\sigma, \hat{x}) \tag{8.6}$$

with $\text{Re } \sigma = -d/2$. It follows, in particular, that the spectrum H^2 acting in $L^2(\mathbf{R}^d)$ is $]-\infty, -d(d-1)/4]$ and that of M_2 is $[Dd^2/4, \infty[$. Denoting $e^{-tJ^2} \equiv k_t$, we obtain

$$\begin{aligned}
& \int f(x) e^{-tM^2(x, x_0)} dx \\
&= \int_{\text{Re } \sigma = -d/2} \frac{d\sigma}{2\pi i} \int_0^\infty dr \int d\hat{x} r^{-\sigma-1} f(r\hat{x}) r_0^\sigma e^{iD\sigma(\sigma+d)} k_{D(d+1)/(d-1)}(r\hat{x}, \hat{x}_0) \\
&= \int_0^\infty \frac{dr}{r} \int d\hat{x} f(r\hat{x}) \frac{1}{\sqrt{4\pi Dt}} e^{-(1/(4Dt))(\ln(r/r_0) - tDd)^2} \\
&\quad \times k_{D(d+1)/(d-1)}(r\hat{x}, \hat{x}_0) \tag{8.7}
\end{aligned}$$

where we have performed the Gaussian integral over σ . The result (7.2) readily follows. Taking $f(x) = \mathcal{C}_L(r) = \mathcal{C}(r/L)$ with \mathcal{C} the rotationally invariant forcing covariance, we obtain by integrating over t the expression for the 2-point function of T at $\gamma = 0$:

$$\begin{aligned}
\overline{\mathcal{F}}_{2,L}(x) &= (M_2^{-1} \mathcal{C}_L)(x) \\
&= \frac{1}{Dd} \left(\int_r^\infty \mathcal{C}(\rho/L) \frac{d\rho}{\rho} + r^{-d} \int_0^r \mathcal{C}(\rho/L) \rho^{d-1} d\rho \right) \tag{8.8}
\end{aligned}$$

Clearly $\overline{\mathcal{F}}_{2,L}(x)$ is smooth for $x \neq 0$ and

$$\overline{\mathcal{F}}_{2,L}(x) \cong -\frac{\mathcal{C}(0)}{Dd} \ln(r/L) \quad \text{for small } r \tag{8.9}$$

$$\overline{\mathcal{F}}_{2,L}(x) \sim (r/L)^{-d} \quad \text{for large } r \tag{8.10}$$

In order to solve Eqs. (2.23) for the higher-point functions of T we need a representation for the Green function M_n^{-1} . This is obtained by relating H^2 and J^2 to the Casimirs \mathcal{H}^2 and \mathcal{J}^2 of the left action of $SL(d)$ and of $SO(d)$ on functions F on $SL(d)$, given by $(\mathcal{L}_g F)(h) = F(g^{-1}h)$, or in the infinitesimal form by $(d/dt)|_{t=0} \mathcal{L}_{e^{tA}} F = A^{\beta\alpha} \mathcal{H}_{\alpha\beta} F$. Note that $\mathcal{H}_{\alpha\beta}$ are skew-adjoint in the regular representation and that

$$(d+1) \mathcal{J}^2 - d\mathcal{H}^2 = -\frac{d+2}{4} \sum_{\alpha, \beta} (\mathcal{H}_{\alpha\beta} - \mathcal{H}_{\beta\alpha})^2 - \frac{d}{4} \sum_{\alpha, \beta} (\mathcal{H}_{\alpha\beta} + \mathcal{H}_{\beta\alpha})^2 \tag{8.11}$$

is a positive elliptic operator in $L^2(dg)$. In particular, it has the heat kernel

$$\mathcal{K}_t(g, h) \equiv e^{-t(D/(d-1))[(d+1) \mathcal{J}^2 - d\mathcal{H}^2]}(g, h) \tag{8.12}$$

satisfying $\int \mathcal{K}_i(g, h) dh = 1$ and $\mathcal{K}_i(g, h) = \mathcal{K}_i(kg, kh) = \mathcal{K}_i(gg', hg')$ for $k \in SO(d)$ and $g' \in SL(d)$. Assign to a translationally invariant function $f(\mathbf{x})$ and to \mathbf{x} a function $F_{\mathbf{x}}(g) = f(g\mathbf{x})$ on $SL(d)$. The linear map $f \mapsto F$ intertwines the two actions of $SL(d)$:

$$(L_g f)(h\mathbf{x}) = (\mathcal{L}_g F_{\mathbf{x}})(h) \tag{8.13}$$

It follows that

$$\int e^{-tM_n}(g\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d'\mathbf{y} = \int \mathcal{K}_i(g, h) f(h\mathbf{x}) dh \tag{8.14}$$

Comparing the above relation to Eq. (7.14) we conclude that

$$p_t(hg^{-1}) = \mathcal{K}_i(g, h) \tag{8.15}$$

Clearly the basic properties (7.12) and (7.13) of the p.d.f. p_t follow. In other words, we may identify g_{t,t_0} as the diffusion process on group $SL(d)$ with the generator equal to $(D/(d-1))[(d+1)\mathcal{J}^2 - d\mathcal{H}^2]$.

Integrating the relation (8.14) over t we infer that

$$\int M_n^{-1}(g\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d'\mathbf{y} = \int \mathcal{G}(g, h) f(h\mathbf{x}) dh \tag{8.16}$$

where \mathcal{G} is the integral kernel of $((D/(d-1))[(d+1)\mathcal{J}^2 - d\mathcal{H}^2])^{-1}$. Applying iteratively identity (8.16) to Eq. (2.23) we end up with the expression

$$\mathcal{F}_{2n}(\mathbf{x}) = \sum_p F_{2n}(\mathbf{u}_p) \tag{8.17}$$

where the sum runs through all ordered pairings $p = (\{i_1, j_1\}, \dots, \{i_n, j_n\})$ of $\{1 \dots 2n\}$, $\mathbf{u}_p = (x_{i_1 j_1}, \dots, x_{i_n j_n})$ and

$$\begin{aligned} F_{2n}(\mathbf{u}) &= \int \prod_{i=1}^n \mathcal{G}(g_{i-1}, g_i) \mathcal{C}(g_i u_i) dg_i \\ &= \int \prod_{i=1}^n \tilde{\mathcal{G}}(g_{i-1}, g_i) \mathcal{C}(g_i u_i) dg_i \end{aligned} \tag{8.18}$$

where $g_0 = e$ and $\tilde{\mathcal{G}}(g, h) = \int_{SO(d)} \mathcal{G}(g, kh) dk$. The last equality follows by substituting $g_1 = k_1 g'_1$, $g_2 = k_1 k_2 g'_2$, $g_n = k_1 \dots k_n g'_n$, and using $\mathcal{G}(kg_1, kg_2) = \mathcal{G}(g_1, g_2)$ and $\mathcal{C}(kx) = \mathcal{C}(x)$.

The final reduction consists of identifying $\tilde{\mathcal{G}}$ with the Green function of the Laplace–Beltrami operator Δ on the homogeneous space $H_d \equiv SL(d)/SO(d)$. By definition, Δ coincides with the Casimir $\frac{1}{2}\mathcal{H}^2$ if we view functions on H_d as functions on $SL(d)$ right-invariant under the action of $SO(d)$. Assign to a function f on H_d the function $g \mapsto \tilde{f}(g) = f(g^{-1})$. Clearly $\tilde{f}(kg) = \tilde{f}(g)$ and $(\mathcal{L}_g f)(h^{-1}) = \tilde{f}(hg) \equiv (\mathcal{R}_g \tilde{f})(h)$, i.e., the map $f \mapsto \tilde{f}$ intertwines the action of $SL(d)$ on the functions on H_d with the right regular action of $SL(d)$. Since the quadratic Casimirs of $SL(d)$ in the left-regular and in the right-regular representations coincide and \mathcal{J}^2 vanishes in the action on \tilde{f} , we infer that

$$-2d(\Delta f)(g^{-1}) = -d(\mathcal{H}^2 \tilde{f})(g) = ([(d+1) \mathcal{J}^2 - d\mathcal{H}^2] \tilde{f})(g) \quad (8.19)$$

and that

$$\int G(g^{-1}, h) f(h) dh = \int \mathcal{G}(g, h) \tilde{f}(h) dh = \int \tilde{\mathcal{G}}(g, h) \tilde{f}(h) dh \quad (8.20)$$

where the function $G(g, h)$ on $H_d \times H_d$ represents the kernel of the operator $(-D'\Delta)^{-1}$ where $D' \equiv (2Dd/(d-1))$. Thus $\tilde{\mathcal{G}}(g, h) = G(g^{-1}, h^{-1})$ and Eq. (8.18) becomes

$$F_{2n}(\mathbf{u}) = \int \prod_{i=1}^n G(g_{i-1}, g_i) \mathcal{C}(g_i^{-1} u_i) dg_i \quad (8.21)$$

Every matrix $g \in SL(d)$ can be uniquely represented as a product (the so called Iwasawa decomposition) $g = nak$ where $k \in SO(d)$, n is upper triangular with 1 on the diagonal and a is diagonal with positive entries. Thus one may parametrize the cosets $gSO(d)$ by na . For $d=2$ we may write $a = \text{diag}(y^{1/2}, y^{-1/2})$, $y > 0$, $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in \mathbf{R}$. The Haar measure dg becomes $dg = y^{-2} dx dy dk$. The homogeneous space H_2 may be identified with the upper half-plane $H = \{z = x + iy \in \mathbf{C} \mid y > 0\}$. The action of $SL(2)$ on H is given by the Möbius transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az + b)/(cz + d)$. Since $ki = i$, the identification maps the coset $gSO(2)$ to $gi = nai$. We shall denote $na \equiv g(z) = \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y \end{pmatrix}$. The $SL(2)$ -invariant measure on H is $dv(z) = y^{-2} dx dy$ and the Laplace–Beltrami operator becomes

$$\Delta = y^2(\partial_y^2 + \partial_x^2) \quad (8.22)$$

The Green function G is given by the explicit expression:

$$G(z, z') = \frac{1}{16 D\pi} \ln \frac{(x-x')^2 + (y+y')^2}{(x-x')^2 + (y-y')^2} \quad (8.23)$$

Eq. (8.21) may now be rewritten as

$$F_{2n}(\mathbf{u}) = \int \prod_{i=1}^n G(z_{i-1}, z_i) \mathcal{C}(g(z_i)^{-1} u_i) dv(z_i) \quad (8.24)$$

with $z_0 = i$. In Appendix B we study the integrals (8.24) in more detail. In particular we show that the leading singularities at coinciding points of the correlation functions of T are given by a Gaussian expression, a sum of products of 2-point functions, confirming the analysis of refs. 1 and 6.

For the dimension $d > 2$ one can proceed analogously. In the Iwasawa decomposition we parametrize n by the off-diagonal entries, x_α , $\alpha = 1, \dots, (d^2 - d)/2$, and write $a = e^\phi$, $\phi = \text{diag}(\phi_1, \dots, \phi_d)$ with $\sum_i \phi_i = 0$. The Haar measure becomes in these variables

$$dg = e^{\sum_{i < j} (\phi_j - \phi_i)} \prod_{i=1}^{d-1} d\phi_i \prod_{\alpha} dx_\alpha dk \quad (8.25)$$

G is (proportional to) the Green function of the Laplace–Beltrami operator Δ on $SL(d)/SO(d)$. Explicitly, for $d = 3$ write $\phi = \frac{1}{2}\alpha \text{diag}(1, -1, 0) + \frac{1}{6}\beta \text{diag}(1, 1, -2)$. Then

$$\Delta = e^{2\alpha} \partial_{x_1}^2 + e^{\alpha+\beta} \partial_{x_2}^2 + e^{\beta-\alpha} (\partial_{x_3} + x_1 \partial_{x_2})^2 + \partial_\alpha^2 + 3\partial_\beta^2 \quad (8.26)$$

and $dg = e^{-\alpha-\beta} d\alpha d\beta dx_1 dx_2 dx_3 dk$. There does not seem to exist a very explicit expression for G in $d > 2$. However, the singular behavior of \mathcal{F}_{2n} can be extracted again, see Appendix B.

Let us end this section by deriving the formula for the Lyapunov exponents of the Lagrangian trajectories, previously found in ref. 12 by path-integral techniques. In ref. 24 it was observed that M_n may be also expressed using the quadratic Casimir of the action of $SL(n-1)$ with the generators

$$G_{ij} = -x_{in}^\alpha \partial_{x_i^\alpha} + \frac{1}{n-1} \delta_{ij} x_{kn}^\alpha \partial_{x_k^\alpha} \quad (8.27)$$

for $1 \leq i, j \leq n-1$. This action corresponds to the natural action of $SL(n-1)$ on the i -index of $x_{in} \equiv x_i - x_n$. Denoting by G^2 the quadratic Casimir $\sum_{i,j} G_{ij} G_{ji}$ and by A the generator of dilations $x_i^\alpha \partial_{x_i^\alpha}$ one obtains⁽²⁴⁾

$$M_n = \frac{D}{d-1} \left[(d+1) J^2 - dG^2 - \frac{d-n+1}{n-1} A(A+d_n) \right] \quad (8.28)$$

Let ρ denote the volume spanned by vectors x_{in} , $i = 1, \dots, n-1$, describing the time t differences of the Lagrangian trajectories starting at time zero from points \mathbf{x}_0 :

$$\rho = \sqrt{\det_{i,j}(x_{in} \cdot x_{jn})} \quad (8.29)$$

We would like to find the p.d.f. of ρ . Note that for a function $f(\rho)$,

$$\begin{aligned} M_n f(\rho) &= -\frac{(d-n+1)D}{(d-1)(n-1)} A(A+d_n) f(\rho) \\ &= -\frac{(n-1)(d-n+1)D}{d-1} \rho \partial_\rho (\rho \partial_\rho + d) f(\rho) \end{aligned} \quad (8.30)$$

This follows from Eq. (8.28) since ρ is $SL(n-1)$ - and $SO(d)$ -invariant. Hence M_n preserves the space of functions $f(\rho)$. Also

$$\int |f(\rho)|^2 \prod_i dx_{in} = \text{const.} \int_0^\infty |f(\rho)|^2 \rho^{d-1} d\rho \quad (8.31)$$

where $\text{const.} = \int \delta(\rho-1) \prod_i dx_{in}$. Hence M_n in the action on $f(\rho)$ is diagonalized by the Mellin transform $f(\rho) \rightarrow \hat{f}(\sigma) = \int_0^\infty \rho^{-\sigma-1} f(\rho) d\rho$ (unitary for $\text{Re } \sigma = -d/2$):

$$\begin{aligned} (M_n f)^\wedge(\sigma) &= -\frac{(n-1)(d-n+1)D}{d-1} \sigma(\sigma+d) \hat{f}(\sigma) \\ &\equiv -D(n) \sigma(\sigma+d) \hat{f}(\sigma) \end{aligned} \quad (8.32)$$

As in Eq. (8.7), we obtain

$$\begin{aligned} &\int f(\rho) P_n(t, \mathbf{x}; 0, \mathbf{x}_0) d'\mathbf{x} \\ &= \int_0^\infty \frac{d\rho}{\rho} f(\rho) \frac{1}{\sqrt{4\pi D(n) t}} e^{-(1/(4D(n) t))(\ln(\rho/\rho^0) - tD(n) d)^2} \end{aligned} \quad (8.33)$$

Hence $(1/t) \ln \rho$ is a Gaussian variable with covariance $2D(n)/t$ tending to zero at large times and with mean $D(n) d$ which, by definition, is the sum of the $(n-1)$ largest Lyapunov exponents describing the effective separation of n Lagrangian trajectories. We infer that the n th Lyapunov exponent is

$$\lambda_n = (D(n+1) - D(n)) d = \frac{1}{2}(d-2n+1) D' \quad (8.34)$$

with d exponents equally spaced and symmetric with respect to the origin, confirming the result of ref. 12.

9. QUADRATURE OF THE $\gamma = 0$ CASE

Let us explicitly construct the stationary state of the passive scalar advected by smooth Gaussian velocity with 2-point function (7.6). Relations (8.17) and (8.21) allow to write a compact expression for the generating function of the $\gamma = 0$ theory:

$$\Phi(\chi) \equiv \langle e^{\int \chi(x) T(x) dx} \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \sum_p \int F_{2n}(\mathbf{u}_p) \prod_i \chi(x_i) dx_i \quad (9.1)$$

Noting that all pairings give the same contribution to the x_i integral and that there are $(2n)!/2^n$ of them we get

$$\Phi(\chi) = \sum_{n=0}^{\infty} (-1)^n \int \prod_{i=1}^n G(g_{i-1}, g_i) V_\chi(g_i) dg_i \quad (9.2)$$

where

$$V_\chi(g) = \frac{1}{2} \int \mathcal{G}(g^{-1}(x-y)) \chi(x) \chi(y) dx dy \quad (9.3)$$

is a non-negative function on H_d bounded by $V_\chi(e) = \frac{1}{2} \mathcal{G}(0) (\int \chi)^2$. Eq. (9.2) may be rewritten in the operator language as ($D' \equiv 2Dd/(d-1)$)

$$\Phi(\chi) = \sum_{n=0}^{\infty} (-1)^n \int \left[\left(\frac{1}{-D'\mathcal{A}} V_\chi \right)^n 1 \right] (e) \quad (9.4)$$

The sum on the right hand side involves the Neuman series for the operator $(-D'\mathcal{A} + V_\chi)^{-1}$, i.e., for the Laplacian on H_d perturbed by a potential. Resumming the series we obtain

$$\Phi(\chi) = 1 - [(-D'\mathcal{A} + V_\chi)^{-1} V_\chi](e) \quad (9.5)$$

which is an explicit expression for the characteristic functional of the stationary state of the $\gamma = 0$ Kraichnan model.

Let us see that the right hand side of Eq. (9.5) makes sense. Using the Feynman–Kac formula expressing the perturbed heat kernel as an expectation $E_g(\cdot)$ with respect to the Brownian motion on H_d with transition amplitudes $e^{-D'\Delta}(g, h)$, starting at time zero at g :

$$e^{-t(-D'\Delta + V_\chi)}(g, h) = E_g(e^{-\int_0^t V_\chi(h(s)) ds} \delta_h(h(t))) \tag{9.6}$$

we infer the bounds

$$0 \leq e^{-t(-D'\Delta + V_\chi)}(g, h) \leq e^{tD'\Delta}(g, h), \tag{9.7}$$

$$0 \leq (-D'\Delta + V_\chi)^{-1}(g, h) \leq G(g, h) \tag{9.8}$$

Since

$$[(-D'\Delta + V_\chi)^{-1} V_\chi](e) = \int (-D'\Delta + V_\chi)^{-1}(e, h) V_\chi(h) dh \tag{9.9}$$

it follows that the latter integral is bounded by the smeared 2-point function

$$\int G(e, h) V_\chi(h) dh = \frac{1}{2} \int \mathcal{F}_2(x_{12}) \chi(x_1) \chi(x_2) dx_1 dx_2 \tag{9.10}$$

which is finite for test functions χ e.g., from the Schwartz space $\mathcal{S}(\mathbf{R}^d)$, see Eq. (8.8).

Φ defines a continuous positive-definite functional on $\mathcal{S}(\mathbf{R}^d)$. The continuity of $\Phi(\chi)$ w.r.t. $\chi \in \mathcal{S}(\mathbf{R}^d)$ is easy: it follows by the Dominated Convergence Theorem from the Feynman–Kac representation of the perturbed Green function:

$$(-D'\Delta + V_\chi)^{-1}(g, h) = \int_0^\infty dt E_g(e^{-\int_0^t V_\chi(h(s)) ds} \delta_h(t)) \tag{9.11}$$

The positive definiteness:

$$\sum_{r,s} \lambda_r \bar{\lambda}_s \Phi(\chi_r - \chi_s) \geq 0 \tag{9.12}$$

is a little bit more complicated. Let us sketch its proof. Define first the positive definite characteristic functional

$$\Phi_t(\chi) = \langle e^{i \int T(t,x) \chi(x) dx} \rangle \tag{9.13}$$

of the time t (quasi-Lagrangian) state of the scalar where $T(t, x) = \int_0^t f(s, g_{t,s}^{-1} x) ds$ is a functional of the forcing f and of $g_{t,s}$. The above

expression for $T(t, x)$ is obtained for the initial condition vanishing at $t_0 = 0$ in Eq. (2.4). The expectation in (9.13) is w.r.t. the Gaussian measure of the forcing and w.r.t. the measure of the diffusion process $g_{t,s}$. It is easy to see that $\int T(t, x) \chi(x) dx$ is square-integrable with respect to these measures. Performing the integration with respect to f , we obtain

$$\Phi_t(\chi) = \langle e^{-\int_0^t V_\chi(g_{t,s}) ds} \rangle \tag{9.14}$$

The remaining expectation over $g_{t,s}$ is easy to calculate by expanding the exponential (the resulting series of expectations converges absolutely for finite t). The result is

$$\Phi_t(\chi) = 1 - \int_0^t [e^{-s(-D'\Delta + V_\chi)} V_\chi](e) ds \tag{9.15}$$

Using the bound (9.7), it is easy to see that $\Phi_t(\chi)$ converge to $\Phi(\chi)$ when $t \rightarrow \infty$. Hence the positive definiteness of Φ . Note that Eq. (9.15) may be rewritten by integration by parts and the Feynman–Kac formula (9.6) as

$$\Phi_t(\chi) = [e^{-t(-D'\Delta + V_\chi)} 1](e) = E_e(e^{-\int_0^t V_\chi(h(s)) ds}) \tag{9.16}$$

which follows also directly from Eq. (9.14) if we notice that the diffusion process $s \mapsto g_{t,t-s}$ on $SL(d)$ projects to the Brownian motion on H_d . The resulting alternative expressions for Φ :

$$\Phi(\chi) = \lim_{t \rightarrow \infty} \int e^{-t(-D'\Delta + V_\chi)}(e, h) dh = E_e(e^{-\int_0^\infty V_\chi(h(s)) ds}) \tag{9.17}$$

relate $\Phi(\chi)$ to the long time behavior of the diffusion on the homogeneous space H_d in the presence of a positive potential V_χ or to the low-energy properties of the Schrödinger operator $-D'\Delta + V_\chi$. They imply that $0 \leq \Phi(\chi) \leq 1$. Expressions (9.17) may also be obtained directly in the Martin–Siggia–Rose (MSR)⁽²⁰⁾ formal functional integral approach.

By Minlos Theorem, the normalized ($\Phi(0) = 1$), continuous, positive-definite functional Φ on $\mathcal{S}(\mathbf{R}^d)$ given by Eqs. (9.5) or (9.17) defines a unique probability measure $d\mu$ on $\mathcal{S}'(\mathbf{R}^d)$ s.t.

$$\Phi(\chi) = \int e^{i \int T(x) \chi(x) dx} d\mu(T) \tag{9.18}$$

$d\mu$ is the stationary state of the Kraichnan model for $\gamma = 0$ alluded to in Sect. 7. It is quite different from the Gibbs measure and quite non-Gaussian

and is, indeed, supported by distributional configurations of the scalar since the correlation functions

$$\mathcal{F}_{2n}(\mathbf{x}) = \int T(x_1) \cdots T(x_{2n}) d\mu(T) \quad (9.19)$$

diverge logarithmically at coinciding points. The measure $d\mu$ contains all the joint p.d.f.'s of smeared scalar values $\int T(x) \chi(x) dx$. In particular, the function $p \mapsto \Phi(p\chi)$, is the Fourier transform of the p.d.f. $p_\chi(\theta)$ of $\int T(x) \chi(x) dx$ whose behavior was studied in ref. 10, see also refs. 6, 8, and 25.

$\Phi(p\chi)$ is a pointwise limit of the finite-time functions $\Phi_t(p\chi)$ which are entire in p . For $\text{Re } p^2 \geq -b^2$, $b > 0$,

$$\begin{aligned} |\Phi_t(p\chi)| &\leq \Phi_t(\pm ib\chi) = E_e(e^{b^2 \int_0^t V_\chi(h(s)) ds}) = \int e^{s(D'A + b^2 V_\chi)}(e, h) dh \\ &= 1 + b^2 \int_0^t ds \int e^{s(D'A + b^2 V_\chi)}(e, h) V_\chi(h) dh \end{aligned} \quad (9.20)$$

The Schrödinger operator $-D'A - b^2 V_\chi$ with a negative potential may develop bound states. The right hand side of the inequality (9.20) grows with t since the expression under the integrals is positive. If $e_b \equiv \inf\{\text{spec}(-D'A - b^2 V_\chi)\} < 0$ then the growth is unbounded since $e^{s(D'A + b^2 V_\chi)}(e, h) \sim e^{-se_b}$ for large s . On the other hand, for $e_b > 0$ the right hand side of (9.20) would be bounded uniformly in t if V_χ were of compact support on H_d . V_χ , however, does not have a compact support as a function on H_d even if \mathcal{C} and χ do (if they do not vanish identically). It is, nevertheless, easy to see from the definition (9.3) that V_χ vanishes at infinity of H_d , i.e., that it gets arbitrarily small outside sufficiently big compact subsets of H_d . This is enough to assure a uniform bound for the right hand side of (9.20) as may be seen by the following argument which separates the behavior at infinity of H_d from that in the interior. Write $V_\chi = V'_\chi + V''_\chi$ where $0 \leq V'_\chi \leq V_\chi$ and V'_χ has a compact support. By the Hölder inequality,

$$E_e(e^{b^2 \int_0^t V_\chi(h(s)) ds}) \leq E_e(e^{(1+\varepsilon)b^2 \int_0^t V'_\chi(h(s)) ds})^{1/(1+\varepsilon)} E_e(e^{((1+\varepsilon)/\varepsilon)b^2 \int_0^t V''_\chi(h(s)) ds})^{\varepsilon/(1+\varepsilon)} \quad (9.21)$$

If we choose ε small so that for $b' = (1+\varepsilon)^{1/2} b$ the relation $e_{b'} > 0$ still holds then the first expectation on the right hand side of inequality (9.21) is bounded uniformly in t (e_b increases with decrease of V_χ). Choose the

support of V'_x so that $((1 + \varepsilon)/\varepsilon) b^2 V''_x \leq v_0 < (Dd^2/4)$ where v_0 is a constant. Then

$$\begin{aligned}
 & E_\varepsilon(e^{((1 + \varepsilon)/\varepsilon) b^2 \int_0^t V''_x(h(s)) ds}) \\
 & \leq 1 + \frac{1 + \varepsilon}{\varepsilon} b^2 \int_0^t ds \int e^{-s(D'A - v_0)}(e, h) V_x(h) dh \\
 & = 1 + \frac{1 + \varepsilon}{2\varepsilon} b^2 \int_0^t ds \int e^{-s(M_2 - v_0)}(x_{12} - y) \\
 & \quad \times \mathcal{C}(y) \chi(x_1) \chi(x_2) dx_1 dx_2 dy \tag{9.22}
 \end{aligned}$$

and the last expression is bounded uniformly in t as may be easily seen from Eq. (8.7).

We infer that $\Phi(p\chi)$, as a limit of uniformly bounded analytic functions, is analytic in p for $\text{Re } p^2 > b_0^2$ but has a singularity at $p = \pm ib_0$ where b_0 is the positive number s.t. $e_{b_0} = 0$. The Cauchy bounds imply now that

$$\left| \frac{d^n}{dp^n} \Phi(p\chi) \right| \leq \frac{\text{const.}(\varepsilon) n!}{1 + |p|^n} \tag{9.23}$$

in any strip $|\text{Im } p| < b_0 - \varepsilon$ for $\varepsilon > 0$. Note also that $\lim_{|p| \rightarrow \infty} \Phi(p\chi) = 0$ by virtue of Eq. (9.17). Since

$$p_\chi(\theta) = \frac{1}{2\pi} \int e^{-i\theta p} \Phi(p\chi) dp \tag{9.24}$$

it is easy to show integrating by parts few times and moving the p -integration contour to $\text{Im } p = \pm(b_0 - \varepsilon)$ that $p_\chi(\theta)$ is smooth except, possibly, at $\theta = 0$ and that

$$p_\chi(\theta) \leq \text{const.}(\varepsilon) e^{-(b_0 - \varepsilon)|\theta|} \tag{9.25}$$

for $|\theta| \geq \mathcal{O}(1)$ and any positive ε . Clearly, the same inequality fails for negative ε since it would imply analyticity of $\Phi(p\chi)$ at $p = \pm ib_0$. *In short:* the p.d.f. $p_\chi(\theta)$ of $\int T(x) \chi(x) dx$ has an exponential decay for large $|\theta|$ with the rate b_0 equal to the value of b at which the ground state of $-D'A - b^2 V_x$ crosses zero energy. Note that the rate b_0 , as related to a bound state energy is not, in general, a semi-classical quantity.

For rotationally invariant χ which, for simplicity, we shall normalize so that $\int \chi = 1$, our operators on H_d reduce to the ones on the double coset space $SO(d) \backslash SL(d) / SO(d)$. This space may be identified with the Cartan algebra of

$SL(d)$ divided by the action of the Weyl group and may be parametrized by the diagonal matrices $\text{diag}(\phi_1, \dots, \phi_d)$ with entries $\phi_1 \leq \dots \leq \phi_d$ and s.t. $\sum_i \phi_i = 0$. In this parametrization, the Schrödinger operator $-D'A + b^2 V_x$ becomes the Calogero–Sutherland Hamiltonian^(22, 28) with a potential:

$$D' \left(-\frac{1}{2} \sum_i \frac{d^2}{d\phi_i^2} - \sum_{i < j} \frac{1}{4 \sinh^2(\phi_j - \phi_i)} + \frac{d(d^2 - 1)}{24} \right) - b^2 V_x(\phi_i) \quad (9.26)$$

acting in $L^2(\prod_i^{d-1} d\phi_i)$. The constant $d(d^2 - 1)/24$, equal to the half length squared of the Weyl vector of $SL(d)$, is the infimum of the spectrum of $-A^{(22)}$ so that $e_b \geq (D'd(d^2 - 1)/24) - b^2 V_x(0)$. Note that, for $d > 2$, $d(d^2 - 1)/24$ is higher than the infimum of the spectrum of $-\frac{1}{2}H^2$ acting in $L^2(\mathbf{R}^d)$ since, as pointed out in the remark after Eq. (8.6), the latter is equal to $d(d-1)/8$. This discrepancy is due to the appearance of different irreducible representations in the decomposition of the actions of $SL(d)$ in $L^2(H_d)$ and in $L^2(\mathbf{R}^d)$ for $d > 2$ and it will play an important role below. When the forcing covariance $\mathcal{C}(r)$ is essentially constant for $r \lesssim L$ and is falling off to zero for $r \gg L$ (e.g., for $\mathcal{C}(r)$ replaced by $\mathcal{C}_L(r) \equiv \mathcal{C}(r/L)$) then the potential $-b^2 V_x$ approaches for $L \rightarrow \infty$ a constant equal to $-\frac{1}{2}b^2 \mathcal{C}(0)$ so that for large L we obtain $\frac{1}{2}b_0^2 \mathcal{C}(0) \cong (D'd(d^2 - 1)/24)$ or

$$b_0 \cong d \sqrt{\frac{D(d+1)}{6\mathcal{C}(0)}} \quad (9.27)$$

Note however that although b_0 stabilizes when $L \rightarrow \infty$, the right hand side of Eq. (9.20) tends to $e^{(1/2)tb^2\mathcal{C}(0)}$ and blows up with t .

The exponential decay of the scalar p.d.f. for $\gamma = 0$ in the isotropic two-dimensional situation was first found in ref. 6, see also ref. 25 for a discussion of the non-isotropic case. The calculation of ref. 6 was extended to higher dimensions in ref. 8. Both calculations were reinterpreted in ref. 10 within the semiclassical approach. Our rigorous result about the decay rate b_0 of $p_x(\theta)$ disagrees for $d > 2$ with the result of ref. 8 and with the instanton calculation of ref. 10. These papers obtain the value $b'_0 = d \sqrt{D/2\mathcal{C}(0)}$ for the decay rate which is smaller than b_0 for $d > 2$. The point is that in refs. 6 and 8 the function $V_x(g)$ of Eq. (9.3) was replaced by $V_x(g) = \frac{1}{2}\mathcal{C}(g^{-1}x)$ with fixed $x \neq 0$. This simplifies the calculation of the expressions of Eq. (9.17) since only the distribution of $g_{i,s}x$ for one x is needed. They become

$$\Phi'_x \equiv E_e(e^{-\int_0^\infty V_x(h(s)) ds}) = \lim_{t \rightarrow \infty} \int_{\mathbf{R}^d} e^{-t(M_2 + (1/2)\mathcal{C})}(x, y) dy \quad (9.28)$$

and lead to the quantum mechanical problem analyzed in refs. 6 and 8. Upon the replacement of V_x by $p^2 V_x$ one obtains a function $\Phi'_x(p)$ whose first singularity off the real axis is at $p = \pm ib$ with b s.t. the ground state of $M_2 - \frac{1}{2}b^2 \mathcal{C}$ crosses zero energy. Since the spectrum of M_2 starts from $Dd^2/4$, see the remark after Eq. (8.6), we indeed obtain, for $\mathcal{C} = \mathcal{C}_L$ and large L , the exponential decay rate b'_0 for the Fourier transform of $\Phi'_x(p)$. The technical reason for the discrepancy with our exact calculation is that $V_x(g)$, unlike its smeared version $V_x(g)$, does not vanish at the infinity of H_d and leads to a more singular behavior of the right hand side of Eq. (9.20). Another way to see it¹⁷ is that Φ'_x is given by a version of Eq. (9.1) with $\int \prod \chi(x_i) dx_i$ omitted and with $F_{2n}(\mathbf{u}_p)$ replaced by the partition-independent contribution $F_{2n}(x, \dots, x)$ corresponding the collinear configuration \mathbf{u}_p giving the most singular behavior when $\mathbf{u}_p \rightarrow 0$ (see Appendix B). The smearing in Eq. (9.1) makes this behavior more regular. Our result persists, however, also if we replace $V_x(g)$ with $\tilde{V}_\psi(g) = \frac{1}{2} \int \mathcal{C}(g^{-1}x) \psi(x) dx$, if ψ and \mathcal{C} are non-negative function from $\mathcal{S}(\mathbf{R}^d)$, since $\tilde{V}_\psi(g)$ still vanishes at infinity. In particular, ψ may vanish around the origin which shows that it is the smearing of collinearity, not the inclusion of coinciding points, which is responsible for the discrepancy between b_0 and b'_0 . The lesson is that the correlation of (non-collinear pairs of) Lagrangian trajectories renders the smeared scalar less intermittent in more than two dimensions and should not be neglected.

It is easy to see that $\Phi(p\chi)$ decays exponentially for large real p . Denote by τ the first exit time of the Brownian motion on H_d from a fixed neighborhood of e . The probability of a given value of τ is bounded by $e^{-\text{const.}/\tau}$. Since $V \equiv \int_0^\infty V_x(h(s)) ds \geq \text{const.} \tau$, the conditional expectation $E_e(e^{-p^2 V} | \tau)$ is bounded by $e^{-\text{const.} p^2 \tau}$. Hence the exponential decay of $E_e(e^{-p^2 V}) \leq \int_0^\infty e^{-\text{const.}(p^2 \tau + 1/\tau)} d\tau$. A more exact description of the decay follows from the path-integral integral representation of the expectation (9.17). The latter implies that the large p behavior of $\Phi(p\chi)$ for real p , unlike the large θ behavior of $p_\chi(\theta)$, is semi-classical:

$$\Phi(p\chi) \sim e^{!p! S(g(\cdot))} \tag{9.29}$$

where $[0, \infty] \mapsto g(s)$ describes a trajectory (instanton) in H_d minimizing the action

$$S(h(\cdot)) = \int_0^\infty \left(\frac{1}{2D'} |h(s)|^2 + V_x(h(s)) \right) ds \tag{9.30}$$

¹⁷ We thank M. Chertkov for suggesting this interpretation.

for fixed initial value $h(0) = e$ (with $|\cdot|^2$ standing for the $SL(d)$ -invariant metric on H_d). This is the same instanton as in the field theoretic MSR analysis of ref. 10. For rotationally invariant χ , the problem reduces to the one on $SO(d) \backslash SL(d) / SO(d)$ with the action

$$S(\phi_i(\cdot)) = \int_0^\infty \left(\frac{1}{2D'} \sum \phi_i(s)^2 + V_\chi(\phi_i(s)) \right) ds \quad (9.31)$$

and with the initial value $\phi_i = 0$. In $d = 2$ the minimal value of S is

$$\frac{1}{\sqrt{D'}} \int_0^\infty \sqrt{V_\chi(\phi)} d\phi > 0 \quad (9.32)$$

where $\phi \equiv \phi_2 - \phi_1$. For $\mathcal{C}(r)$ approximately constant up to $r \cong L$, $V_\chi(\phi)$ is approximately constant up to $\frac{1}{2}\phi \cong \ln L$ and then it decays to zero like $\sim e^{-\phi/2}$. Consequently, the exponential decay rate of $\Phi(p\chi)$ is approximately $\ln L \sqrt{\mathcal{C}(0)/2D}$ for large L , in agreement with refs. 6 and 10. For $d > 2$ and large L the minimum of S is attained on the trajectory which in the region of constant potential goes in the direction $\sqrt{1/d(d-1)}$ $(-1, \dots, -1, d-1)$ and the value of the action is again $\cong \ln L \sqrt{\mathcal{C}(0)/2D}$ up to lower order terms, as pointed out in refs. 8 and 10. The exponential decay of $\Phi(p\chi)$ implies that $p_\chi(\theta)$ is smooth also at zero.

10. CONCLUSIONS

In this paper we have analyzed the stochastic dynamics of Lagrangian trajectories for Gaussian, time-decorrelated random velocity fields considered in the Kraichnan model of passive advection. We found that the dynamics is characterized by two related phenomena. First, the Lagrangian trajectories loose in the limit of high Reynolds numbers the deterministic sense for a fixed velocity realization due to their sensitive dependence on initial conditions. Second, their relative stochastic dynamics is dominated by slow resonance-type modes. The slow modes determine the average characteristics of the spread of Lagrangian trajectories responsible for the loss of their deterministic character. Both phenomena were essentially due to nonsmoothness of the typical velocities signaled by fractional Hölder exponents in their spatial dependence. Since the turbulent velocities are nonsmooth in the limit of high Reynolds numbers, we expect the two phenomena to persist for more realistic velocity ensembles and to continue to be responsible for the anomalous scaling, at least for that of passive

scalars. For the spatially smooth velocities, we calculated the Lyapunov exponents describing the sensitive dependence of the Lagrangian trajectories on initial conditions for distances smaller than the viscous scale. Using harmonic analysis on the symmetric spaces $SL(d)/SO(d)$ we also obtained in this case an explicit form of the characteristic functional of the stationary state of the passive scalar and exhibited an exponential decay of the p.d.f.'s of smeared values of the scalar relating the decay rate to the properties of the ground state of the Calogero–Sutherland Schrödinger operator with a potential.

APPENDIX A

We shall make explicit the structural result of Sect. 6 for the heat kernel $e^{-tM_2(x, x_0)}$. In the angular momentum $l = 0, 1, \dots$ sector,

$$M_2 \equiv M_2(l) = -\frac{D}{r^{d-1}} \partial_r r^{d+1-\gamma} \partial_r + \frac{D(d+1-\gamma)}{d-1} l(d-2+l) r^{-\gamma} \tag{A.1}$$

which is a positive operator in $L^2(]0, \infty[, r^{d-1} dr)$. The generalized eigenfunction of $M_2(l)$ corresponding to eigenvalue $E \geq 0$ involves the Bessel function

$$\varphi_E(r) = r^{\gamma-d/2} J_{\nu_l} \left(\frac{2\sqrt{E/D}}{\gamma} r^{\gamma/2} \right) \tag{A.2}$$

where

$$\nu_l = \frac{1}{\gamma} \sqrt{(d-\gamma)^2 + 4 \frac{d+1-\gamma}{d-1} l(d-2+l)} \tag{A.3}$$

The spectral decomposition of $M_2(l)$ has the form

$$M_2(l) = \int E |\varphi_E\rangle \langle \varphi_E| d\nu(E) \tag{A.4}$$

Since

$$(\mathcal{U}_s \varphi_E)(r) \equiv e^{sd/2} \varphi_E(e^s r) = e^{(\gamma/2)s} \varphi_{e^{\gamma s} E}(r) \tag{A.5}$$

we infer that

$$\mathcal{U}_s M_2(l) \mathcal{U}_s^{-1} = \int E |\varphi_E\rangle \langle \varphi_E| d\nu(e^{\gamma s} E) \tag{A.6}$$

Since, on the other hand, $\mathcal{U}_s M_2 \mathcal{U}_s^{-1} = e^{\gamma s} M_2$, see Eq. (6.3), it follows that

$$dv(e^{\gamma s} E) = e^{\gamma s} dv(E) \quad (\text{A.7})$$

i.e., that $dv(E) = c dE$ for some positive constant c . Hence

$$c \int |\varphi_E\rangle \langle \varphi_E| dE = I \quad (\text{A.8})$$

and for

$$\hat{f}(E) = \sqrt{c} \int_0^\infty \frac{\overline{\varphi_E(r)} f(r)}{\varphi_E(r)} r^{d-1} dr \quad (\text{A.9})$$

we obtain $\int_0^\infty |\hat{f}(E)|^2 dE = \int_0^\infty |f(r)|^2 r^{d-1} dr$. Substituting $E = e^{\gamma u}$, we shall define

$$(\mathcal{V}_1 f)(u) = \sqrt{\gamma} e^{(\gamma/2)u} \hat{f}(e^{\gamma u}) \quad (\text{A.10})$$

$\mathcal{V}_1: L^2(]0, \infty[, r^{d-1} dr) \rightarrow L^2(\mathbf{R}, du)$ is a unitary operator. Besides,

$$(\mathcal{V}_1 M_2(l) f)(u) = e^{\gamma u} (\mathcal{V}_1 f)(u) \quad (\text{A.11})$$

Let $\mathcal{V}_2: L^2(]0, \infty[, r^{d-1} dr) \rightarrow L^2(\mathbf{R}, du)$ be another unitary operator defined by

$$(\mathcal{V}_2 f)(u) = e^{-(d/2)u} f(e^{-u}) \quad (\text{A.12})$$

Note that

$$(\mathcal{V}_i \mathcal{U}_s f)(u) = (\mathcal{V}_i f)(u - s), \quad i = 1, 2 \quad (\text{A.13})$$

so that $U_2 = \mathcal{V}_1^{-1} \mathcal{V}_2$ commutes with \mathcal{U}_s . Besides,

$$M_2(l) = U_2 r^{-\gamma} U_2^{-1} \quad (\text{A.14})$$

as follows from Eq. (A.11). This is the relation (6.8) for $n=2$. Since, explicitly,

$$(\mathcal{V}_1 f)(u) = \sqrt{\gamma c} \int e^{(\gamma/2)(u-u')} J_{\nu_i} \left(\frac{2}{\gamma \sqrt{D}} e^{(\gamma/2)(u-u')} \right) (\mathcal{V}_2 f)(u') du' \quad (\text{A.15})$$

and the Mellin transform is the composition of \mathcal{V}_2 and the Fourier transform, we obtain for $\text{Re } \sigma = d/2$

$$\begin{aligned} \hat{U}_2(\sigma)^{-1} &= \sqrt{\gamma c} \int e^{(d/2 + \sigma)u} e^{(\gamma/2)u} J_{\nu_l} \left(\frac{2}{\gamma \sqrt{D}} e^{(\gamma/2)u} \right) du \\ &= \sqrt{\gamma c D} \left(\frac{\gamma}{2} \sqrt{D} \right)^{(d+2\sigma)/\gamma} \int x^{(d+2\sigma)/\gamma} J_{\nu_l}(x) dx \\ &= \sqrt{\gamma c D} (\gamma \sqrt{D})^{(d+2\sigma)/\gamma} \frac{\Gamma(1/2(1 + \nu_l + (d+2\sigma)/\gamma))}{\Gamma(1/2(1 + \nu_l - (d+2\sigma)/\gamma))} \end{aligned} \quad (\text{A.16})$$

The unitarity implies that $\gamma c D = 1$ so that, finally,

$$\hat{U}_2(\sigma) = (\gamma \sqrt{D})^{-(d+2\sigma)/\gamma} \frac{\Gamma(1/2(1 + \nu_l - (d+2\sigma)/\gamma))}{\Gamma(1/2(1 + \nu_l + (d+2\sigma)/\gamma))} \quad (\text{A.17})$$

The right hand side has a meromorphic continuation to the complex plane of σ with poles at

$$\sigma_{l,p} = -\frac{d-\gamma}{2} + \frac{\gamma}{2} \nu_l + \gamma p \quad (\text{A.18})$$

for $p=0, 1, \dots$. Since the true (more regular at the origin) zero mode of $M_2(l)$ occurs at scaling dimension $\sigma_{l,0}$, this is exactly the analytic structure predicted for $\hat{U}_n(\sigma)$. The function $r^{\sigma,p}$ (multiplied by an angular term) represents a slow 2-point mode in the angular momentum l sector.

APPENDIX B

Let us briefly consider the convergence properties of the integrals (8.24). Let $k_i \in SO(2)$ be rotation matrices s.t. $u_i = k_i(r_i, 0)$ where $r_i = |u_i|$. We have $g(z_i)^{-1} k_i = (k_i^{-1} g(z_i))^{-1} = k'_i g(k_i^{-1} z_i)^{-1}$ for some $k'_i \in SO(2)$. Denoting $\mathcal{C}(k(r, 0)) \equiv \mathcal{C}(r)$, observing that $|g(z_i)^{-1}(r_i, 0)| = r_i y_i^{-1/2}$ and using the $SL(2)$ invariance of $dv(z_i)$, we obtain

$$F_{2n}(\mathbf{u}) = \int G(i, z_i) G(\kappa_1 z_1, z_2) \cdots G(\kappa_{n-1} z_{n-1}, z_n) \prod_i \mathcal{C}(r_i y_i^{-1/2}) dv(z_i) \quad (\text{B.1})$$

where $\kappa_i = \kappa_{i+1}^{-1} k_i$ is the rotation by the angle between u_{i+1} and u_i . We shall study the behavior of F_{2n} as r_i tend to 0. The following is a useful relation:

$$\int G(z, z') dx' = \frac{1}{4D} (y\theta(y' - y) + y'\theta(y - y')) \equiv G_0(y, y') \quad (\text{B.2})$$

For the 4-point function, noting that $\text{Im}(\kappa_1 z_1) = \gamma_1 y_1$ where

$$\gamma_1 = [(x_1 \sin \vartheta + \cos \vartheta)^2 + y_1^2 \sin^2 \vartheta]^{-1} \quad (\text{B.3})$$

ϑ being the angle between u_2 and u_1 , we obtain

$$F_4(u_1, u_2) = \int G(i, z_1) G_0(\gamma_1 y_1, y_2) \mathcal{C}(r_1 y_1^{-1/2}) \mathcal{C}(r_2 y_2^{-1/2}) y_2^{-2} dy_2 dv(z_1). \quad (\text{B.4})$$

Consider first the case $\vartheta = 0$ i.e., $\gamma_1 = 1$. Then

$$\begin{aligned} F_4(u_1, u_2) &= \frac{1}{(4D)^2} \int (\theta(y_1 - 1) + y_1 \theta(1 - y_1))(y_1 \theta(y_2 - y_1) \\ &\quad + y_2 \theta(y_1 - y_2)) \\ &\quad \times \mathcal{C}(r_1 y_1^{-1/2}) \mathcal{C}(r_2 y_2^{-1/2}) y_1^{-2} dy_1 y_2^{-2} dy_2 \end{aligned} \quad (\text{B.5})$$

Since $\mathcal{C} = \mathcal{C}_L$ has rapid decay at infinity, the integrals are effectively cut to $y_i > (r_i/L)^2$ and produce logarithms of (r_i/L) as these ratios tend to zero. The most singular contribution is from $y_1 \theta(1 - y_1) y_2 \theta(y_1 - y_2)$ term which yields $4 \ln(r_1/L) \ln(r_2/L) - 2(\ln(r_1/L))^2$ if $r_1 > r_2$ and $2(\ln(r_2/L))^2$ if $r_2 > r_1$. Thus

$$\begin{aligned} &F_{4,L}(u_1, u_2) + F_4(u_2, u_1) \\ &= \left(\frac{\mathcal{C}(0)}{2D} \right)^2 \ln(r_1/L) \ln(r_2/L) + \text{less singular} \end{aligned} \quad (\text{B.6})$$

for $\vartheta = 0$.

For $\vartheta \neq 0$, the y_2 integral yields

$$\begin{aligned} F_{4,L}(u_1, u_2) &= \frac{1}{8\pi} \int \ln \frac{x_1^2 + (y_1 + 1)^2}{x_1^2 + (y_1 - 1)^2} \mathcal{C}_L(r_1 y_1^{-1/2}) [(\ln(\gamma_1) + \ln(y_1 r_2^{-2} L^2)) \\ &\quad \times \vartheta(\gamma_1 y_1 - r_2^2/L^2) + B] y_1^{-2} dy_1 dx_1 \end{aligned} \quad (\text{B.7})$$

where B is bounded. For the $\ln(\gamma_1)$ term the only singularity is at y_1 small and this term is bounded by $\text{const.} |\ln(r_1/L)|$. The rest has same leading singularity as the $\vartheta=0$ calculation. Thus recognizing in (B.6) the 2-point singularities (8.9), we infer that the leading singularity of the 4-point function is Gaussian, the sum of products of 2-point functions.

The analysis of the general correlation is similar though tedious. When all the points are on the same line i.e., all the angles are zero, we can do all the x_i integrals by (B.2). Most singular contribution is the one where all $G_0(y_{i-1}, y_i)$ are replaced by $1/(4D) y_i \theta(y_{i-1} - y_i)$. Summing over the permutations of the u_i yields

$$\begin{aligned} \sum_{\pi} F_{2n}(\mathbf{u}_{\pi}) &= \prod_i \frac{1}{4D} \int_0^1 \mathcal{C}_L(r_i y^{-1/2}) y^{-1} dy + \dots \\ &= \prod_i -\frac{\mathcal{C}(0)}{2D} \ln(r_i/L) + \dots \end{aligned} \tag{B.8}$$

where \dots is less singular. Non-zero angles give again subleading contributions. The subsequent sum over unordered pairings gives the Gaussian expression for the leading short-distance singularity of $\mathcal{F}_{2n, L}(\mathbf{x})$ in terms of the singular contributions to the 2-point function, in agreement with the observations of refs. 1 and 6.

Finally, for $d > 2$, to extract the leading singularity some bounds for G are needed. Let us here note only that for $d=3$ if all the κ_i are identity, then the x integrals can again be done and the result is that G gets replaced by G_0 where

$$G_0 = \text{const.} (-y_1^2 \partial_{y_1}^2 - 3y_2^2 \partial_{y_2}^2)^{-1} \tag{B.9}$$

on $L^2((y_1 y_2)^{-2} dy_1 dy_2)$ (we have put $y_1 = e^\alpha, y_2 = e^\beta$ in (8.26)). The behavior of G_0 near $y_i=0$ is calculable and the leading singularity can again be shown to be given by products of 2-point functions. Using the Mellin transform we may write

$$\begin{aligned} (G_0 F)(y'_1, y'_2) &= \text{const.} \int_0^\infty \frac{dt}{t} e^{-t} \int \frac{dy_1 dy_2}{y_1 y_2} \\ &\quad \times \left(\frac{y'_1 y'_2}{y_1 y_2} \right)^{1/2} e^{-1/(4t)(\log(y_1/y'_1))^2 - 1/(12t)(\log(y_2/y'_2))^2} \\ &\quad \times F(y_1, y_2) \end{aligned} \tag{B.10}$$

$\mathcal{C}(ry^{-1/2})$ in (B.1) is replaced by $\mathcal{C}(ry_1^{-1/2}y_2^{-1/6})$. Hence, let $\rho = y_1^{1/2}y_2^{1/6}$ and let $F(y_1, y_2) = f(\rho)$. Then $(G_0 F)(y'_1, y'_2) = \tilde{f}(\rho')$ with

$$\tilde{f}(\rho') = \text{const.} \int_0^\infty \frac{dt}{t} e^{-t} \int du dv f(e^{-u-v}\rho') e^{u+3v} e^{-(u^2/t)-(3v^2/t)} \quad (\text{B.11})$$

which after performing the $u-v$ and the t integrals becomes $\int g(\rho', \rho) f(\rho) d\rho$ with

$$g(\rho', \rho) = \text{const.} \left[\frac{1}{\rho} \theta(\rho' - \rho) + \frac{\rho'^3}{\rho^4} \theta(\rho - \rho') \right] \quad (\text{B.12})$$

We may then write F_{2n} in the form

$$F_{2n}(\mathbf{u}) = \int g(1, \rho_1) \cdots g(\rho_{n-1}, \rho_n) \prod_{i=1}^n \mathcal{C}(r_i \rho_i^{-1}) d\rho_i \quad (\text{B.13})$$

and the analysis of the singularities goes on as in the $d=2$ case.

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